UNIT-I

PARTIAL DIFFERENTIAL EQUATIONS

Partial Differential Equations:

Partial differential equations are those equation which contain partial differential coefficients, independ variables and dependent variables.

The independent variables will be denoted x and y and the dependent variable by z. The partial differential coefficients are denoted as follows:

$$\frac{\partial z}{\partial x} = p \quad , \quad \frac{\partial z}{\partial y} = q$$

$$\frac{\partial^2 z}{\partial x^2} = r \quad , \quad \frac{\partial^2 z}{\partial y^2} = t \quad , \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s$$

Order of partial Differential Equations:

Order of partial differential equations is the same as that of the order of the highest differential coefficient in it.

Method of Forming partial Differential Equation:

A partial differential equation is formed by two methods.

1. By eliminating arbitrary constants.
2. By eliminating arbitrary functions.

Method of Eliminating of Arbitrary constant:

1: Form a partial differential equation from

$$x^2 + y^2 + (z-c)^2 = a^2.$$

Solution: The given equation is;

$$x^2 + y^2 + (z-c)^2 = a^2 \quad \text{——— (1)}$$

Equation (1) contains two arbitrary constants a and c.

Differentiating (1) (partially) with respect to x, we get

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$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$\therefore x + (z-c)p = 0$

$\therefore x = -(z-c)p$ $\qquad\qquad \because p = \frac{\partial z}{\partial x}$

$\therefore z - c = -\frac{x}{p}$ —— ②

Now differentiating ① partially with respect to y, we get

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$\therefore y + (z-c)q = 0$ $\qquad\qquad \because \frac{\partial z}{\partial y} = q$

$\therefore y = -(z-c)q$

$\therefore (z-c) = -\frac{y}{q}$ —— ③

Let us eliminate z from ② and ③, we get

$$-\frac{x}{p} = -\frac{y}{q}$$

$\therefore xq = yp$

$\therefore yp = xq$

$\therefore yp - xq = 0$

Which is required partial differential equation.

<u>Method of Elimination of Arbitrary Function</u>

Ex. 2: Form the partial differential equation from
$$z = f(x^2 - y^2).$$

Solution: The given equation is

$$z = f(x^2 - y^2). \text{ —— ①}$$

Differentiating ① with respect to x and y, we get

$$\frac{\partial z}{\partial x} = p = f'(x^2 - y^2) \cdot 2x.$$

$$\frac{p}{2x} = f'(x^2 - y^2) \text{ —— ②}$$

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$$\frac{\partial z}{\partial y} = q = f'(x^2-y^2)(-2y)$$

$$\frac{q}{-2y} = f'(x^2-y^2) \quad \text{——} \ (3)$$

From equations (2) and (3), we get

$$\frac{p}{2x} = \frac{q}{-2y}$$

$\therefore \quad -yp = xq$

$\therefore \quad xq + yp = 0 \quad$ or $\quad yp + xq = 0$

which is required partial differential equation.

Exercise 9.1

- Form the partial differential equation

$$z = (x+a)(y+b)$$

ution: The given equation is

$$z = (x+a)(y+b) \quad \text{——} \ (1)$$

Differentiating (1) with respect to x and y, we get

$$\frac{\partial z}{\partial x} = p = (y+b) \quad \text{——} \ (2)$$

$$\frac{\partial z}{\partial y} = q = (x+a) \quad \text{——} \ (3)$$

Hence $pq = (y+b)(x+a) = z$ By (1)

Hence $pq = z$

which is required partial differential equation.

- Form the partial differential equation.

$$(x-h)^2 + (y-k)^2 + z^2 = a^2$$

ution: The given equation is

$$(x-h)^2 + (y-k)^2 + z^2 = a^2 \quad \text{——} \ (1)$$

Differentiating (1) with respect to x and y, we get

$$2(x-h) + 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$(x-h) + zp = 0$$

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$$(x-h) = -zp \quad \text{--- (2)}$$

and $$2(y-k) + 2z.\frac{\partial z}{\partial y} = 0$$

$$(y-k) + zq = 0$$

$$(y-k) = -zq \quad \text{--- (3)}$$

putting the values of $(x-h)$ and $(y-k)$ in (1), we get

$$(-zp)^2 + (-zq)^2 + z^2 = a^2$$

$$z^2p^2 + z^2q^2 + z^2 = a^2$$

$$z^2(p^2+q^2+1) = a^2$$

which is the required partial differential equation

3. Form the partial differential equation

$$2z = (ax+y)^2 + b.$$

Solution: The given equation is

$$2z = (ax+y)^2 + b \quad \text{--- (1)}$$

Differentiating (1) with respect to x and y, we get

$$2\frac{\partial z}{\partial x} = 2(ax+y)\cdot a$$

$$p = a(ax+y)$$

$$\frac{p}{a} = (ax+y) \quad \text{--- (2)}$$

and $$2\cdot\frac{\partial z}{\partial y} = 2(ax+y)\cdot 1$$

$$q = (ax+y) \quad \text{--- (3)}$$

$$(2) \Rightarrow px = ax(ax+y)$$

$$(3) \Rightarrow qy = y(ax+y)$$

Hence, $$px+qy = ax(ax+y) + y(ax+y)$$

$$= (ax+y)(ax+y)$$

$$= (ax+y)^2$$

Hence, $(px+qy) = q^2$ (Due to (3))

which is the required partial differential equation.

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Form the partial differential equation

$$ax^2+by^2+z^2=1$$

lution: The given equation is

$$ax^2+by^2+z^2=1 \quad \text{——} ①$$

Differentiating ① with respect to x and y, we get

$$2ax+2z\cdot\frac{\partial z}{\partial x}=0$$

$$ax+zp=0$$

$$ax=-zp$$

$$ax^2=-zpx \quad \text{——} ②$$

and
$$2by+2z\frac{\partial z}{\partial y}=0$$

$$by+zq=0$$

$$by=-zq$$

$$by^2=-zyq \quad \text{——} ③$$

$\therefore$ ② + ③ $\Rightarrow$ $ax^2+by^2=-z(px+qy)$

$\Rightarrow$ $1-z^2=-z(px+qy)$ Due to ①

$\Rightarrow$ $z(px+qy)=z^2-1$

which is required partial differential equation.

: Form the partial differential equation.

$$x^2+y^2=(z-c)^2\cdot\tan^2\alpha.$$

lution: The given equation is

$$x^2+y^2=(z-c)^2\tan^2\alpha \quad \text{——} ①$$

Differentiating ① with respect to x and y, we get

$$2x=2(z-c)\tan^2\alpha\ \frac{\partial z}{\partial x}$$

$$x=(z-c)\tan^2\alpha\cdot p$$

$$\frac{x}{p}=(z-c)\tan^2\alpha. \quad \text{——} ②$$

and
$$2y=2(z-c)\tan^2\alpha\cdot\frac{\partial z}{\partial y}$$

$$y=(z-c)\tan^2\alpha\cdot q$$

$$\frac{y}{q}=(z-c)\tan^2\alpha. \quad \text{——} ③$$

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from equations (2) and (3), we get

$$\frac{x}{p} = \frac{y}{q} \Rightarrow xq = py \quad \text{or} \quad yp - xq = 0$$

which is required partial differential equation

6. Form the partial differential equation.

$$z = f(x^2+y^2)$$

Solution: The given equation is

$$z = f(x^2+y^2) \quad \text{——} (1)$$

Differentiating (1) with respect to x and y, we get

$$\frac{\partial z}{\partial x} = f'(x^2+y^2)\cdot 2x$$

$$\frac{p}{x} = 2f'(x^2+y^2) \quad \text{——} (2)$$

and
$$\frac{\partial z}{\partial y} = f'(x^2+y^2)\cdot 2y$$

$$\frac{q}{y} = 2f'(x^2+y^2) \quad \text{——} (3)$$

from equation (2) and (3), we get

$$\frac{p}{x} = \frac{q}{y} \Rightarrow yp = qx \Rightarrow yp - xq = 0$$

which is required partial differential equation.

7. Form the partial differential equation.

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Solution: The given equation is

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{——} (1)$$

Differentiating (1) with respect to x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow p = \frac{x}{a^2} \Rightarrow px = \frac{x^2}{a^2} \quad \text{——} (2)$$

and
$$2\frac{\partial z}{\partial y} = \frac{2y}{b^2} \Rightarrow q = \frac{y}{b^2} \Rightarrow qy = \frac{y^2}{b^2} \quad \text{——} (3)$$

(2) + (3) $\Rightarrow$ $px + qy = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ Due to (1)

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$$\Rightarrow 2z = xp + yq.$$

which is required partial differential equation.

Form the partial differential equation.

$$f(x+y+z,\ x^2+y^2+z^2) = 0$$

Solution: The given equation is

$$f(x+y+z,\ x^2+y^2+z^2) = 0 \quad \text{——} (1)$$

Let $u = x+y+z$, $v = x^2+y^2+z^2$ —— (2)

$\therefore (1) \Rightarrow f(u,v) = 0$ —— (3)

where f is an arbitrary function and u, v are function of x, y, z.

Differentiating (3) with respect to x and y, we get

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] = 0$$

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right] = 0$$

$$\frac{\partial f}{\partial u}\left[1 + p.1\right] + \frac{\partial f}{\partial v}\left[2x + p(2z)\right] = 0 \qquad \text{Due to (2)}$$

$$(1+p)\frac{\partial f}{\partial u} + (2x+2pz)\frac{\partial f}{\partial v} = 0 \quad \text{——} (4)$$

and

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right] = 0$$

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right] = 0$$

$$\frac{\partial f}{\partial u}\left[1 + q\cdot 1\right] + \frac{\partial f}{\partial v}\left[2y + q\cdot 2z\right] = 0 \qquad \text{Due to (2)}$$

$$(1+q)\frac{\partial f}{\partial u} + (2y+2qz)\frac{\partial f}{\partial v} = 0 \quad \text{——} (5)$$

Eliminating $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ from equations (4) and (5), we get

$$(1+p)(2y+2qz) - (2x+2pz)(1+q) = 0.$$

$\because$ If $ax+by=0$ and $cx+dy=0$ } Eliminating x and y, we get $ad-bc=0$.

$$2y + 2qz + 2py + 2pqz - 2x - 2xq - 2pz - 2zpq = 0$$

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$$2(y-x)+(z-x)2q+(y-z)2p=0$$
$$y-x+(z-x)q+(y-z)p=0$$
$$(y-z)p+(z-x)q=x-y$$

Solution of Equation By Direct Integration:

Ex. 3. Solve
$$\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$$

Solution: Given equation is
$$\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y) \quad \text{—} (1)$$

Integrating with respect to x, we get
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}\sin(2x+3y)+f(y),$$

where the constant with regard to x being possibly a function of y.

Integrating with respect to x, we get
$$\frac{\partial z}{\partial y} = -\frac{1}{4}\cos(2x+3y)+f(y).x+g(y)$$

where the constant with regard to x being possibly a function of y.

Integrating with respec to y, we get
$$z = -\frac{1}{12}\sin(2x+3y)+x\int f(y)\,dy+\int g(y)\,dy + c$$

where the constant with regard to y being possibly a function of x.
$$= -\frac{1}{12}\sin(2x+3y)+x\phi_1(y)+\phi_2(y)+\phi_3(x)$$

where $\phi_1(x)=\int f(y)\,dy$, $\phi_2(y)=\int g(y)\,dy$.

Ex. 4 Solve
$$\frac{\partial^2 z}{\partial x \partial y} = x^2 y$$

subject to the condition $z(x,0)=x^2$ and $z(1,y)=\cos y$.

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ution: The given equation is

$$\frac{\partial^2 z}{\partial x \partial y} = x^2 y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2 y.$$

Integrating with respect to x, we get

$$\frac{\partial z}{\partial y} = \frac{x^3}{3} y + f(y)$$

Integrating with respect to y, we get

$$z = \frac{x^3}{3} \cdot \frac{y^2}{2} + \int f(y)\,dy + g(x).$$

$$z(x,y) = \frac{x^3 y^2}{6} + F(y) + g(x), \text{ where } \int f(y)\,dy = F(y) \quad \text{--- (1)}$$

condition 1 : putting $z = x^2$ and $y=0$ in (1), we get

$$z(x,0) = x^2 = 0 + F(0) + g(x)$$

$$x^2 - F(0) = g(x).$$

putting the value of $g(x)$ in (1), we get

$$z = \frac{x^3 y^2}{6} + F(y) + x^2 - F(0) \quad \text{--- (2)}$$

condition 2 : putting $z = z(1,y) = \cos y$ and $x=1$ in (2), we get

$$z(1,y) = \cos y = \frac{y^2}{6} + F(y) + 1 - F(0)$$

$$\cos y - \frac{y^2}{6} + F(0) - 1 = F(y)$$

putting the value of $F(y)$ in (2), we get

$$z = \frac{x^3 y^2}{6} + \cos y - \frac{y^2}{6} - 1 + F(0) + x^2 - F(0)$$

$$= \frac{x^3 y^2}{6} + \cos y - \frac{y^2}{6} - 1 + x^2$$

5 Solve

$$\frac{\partial^2 z}{\partial y^2} = z,$$

if $y=0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$.

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Solution: The given equation is

$$\frac{\partial^2 z}{\partial y^2} = z$$

$$\frac{\partial^2 z}{\partial y^2} - z = 0$$

$$\left(\frac{\partial^2}{\partial y^2} - 1\right) z = 0$$

$\therefore$ Auxiliary equation is

$$m^2 - 1 = 0 \quad , \quad \frac{\partial}{\partial y} = m$$

$$m^2 = 1$$

$$m = \pm 1$$

$\therefore$ $z = C_1 e^y + C_2 e^{-y}$, if z is a function of y alone.

or $z = A\cosh y + B\sinh y$ or $z = A\sinh y + B\cosh y$

$$\because \; A\left(\frac{e^y + e^{-y}}{2}\right) + B\left(\frac{e^y - e^{-y}}{2}\right) = A\cosh y + B\sinh y$$

$$\therefore \; A\cosh y + B\sinh y = \left(\frac{A}{2} + \frac{B}{2}\right) e^y + \left(\frac{A}{2} - \frac{B}{2}\right) e^{-y}$$

$$= c_1 e^y + c_2 e^{-y}$$

where $c_1 = \frac{A}{2} + \frac{B}{2}$, $c_2 = \frac{A}{2} - \frac{B}{2}$

and A, B are constants.

Since z is a function of x and y, A and B are the function of x.

Then $z = \sinh y \cdot f(x) + \cosh y \cdot \phi(x)$ —— ①

on putting $y = 0$ and $z = e^x$ in ①, we get

$$e^x = \phi(x)$$

putting value of $\phi(x)$ in ①, we get

$$z = \sinh y \cdot f(x) + \cosh y \cdot e^x \quad \text{—— ②}$$

Differentiating ② with respect to y, we get

$$\frac{\partial z}{\partial y} = \cosh y \cdot f(x) + \sinh y \; e^x \quad \text{—— ③}$$

on putting $y = 0$ and $\frac{\partial z}{\partial y} = e^{-x}$ in ③, we get

$$e^{-x} = f(x)$$

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putting value of $f(x)$ in ②, we get

$$Z = e^{-x}\sinh y + e^{x}\cosh y$$

which is the required solution.

Exercise 9.2

Solve the following:

$$\frac{\partial^2 z}{\partial x \partial y} = xy^2$$

Solution: The given equation is

$$\frac{\partial^2 z}{\partial x \partial y} = xy^2 \quad \text{——} ①$$

Integrating with respect to x, we get

$$\frac{\partial z}{\partial y} = \frac{x^2}{2}y^2 + F(y)$$

Integrating with respect to y, we get

$$z = \frac{x^2}{2}\cdot\frac{y^3}{3} + \int F(y)\,dy + \phi(x).$$

$$= \frac{x^2y^3}{6} + f(y) + \phi(x), \text{ where } \int F(y)\,dy = f(y)$$

f, ϕ are two arbitrary functions.
which is required solution of ①.

Solve

$$\frac{\partial^2 z}{\partial x \partial y} = e^y \cos x$$

Solution: The given equation is

$$\frac{\partial^2 z}{\partial x \partial y} = e^y \cos x \quad \text{——} ①$$

Integrating ① with respect to x, we get

$$\frac{\partial z}{\partial y} = e^y \sin x + F(y) \quad \text{——} ②$$

Integrating ② with respect to y, we get

$$z = e^y.\sin x + \int F(y)\,dy + \phi(x)$$

$$= e^y.\sin x + f(y) + \phi(x), \quad f(y) = \int F(y)\,dy.$$

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where f, ϕ are two arbitrary functions, which is the required solution of ①.

3. Solve
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{y}{x} + 2$$

Solution: The given equation is
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{y}{x} + 2 \quad \text{—— ①}$$

Integrating ① with respect to x, we get
$$\frac{\partial z}{\partial y} = y \int \frac{dx}{x} + 2\int dx + F(y)$$
$$= y \log x + 2x + F(y) \quad \text{—— ②}$$

Integrating ② with respect to y, we get
$$z = \frac{y^2}{2} \log x + 2x \int dy + \int F(y)\,dy + \phi(x)$$
$$= \frac{y^2 \log x}{2} + 2xy + f(y) + \phi(x),$$

$$f(y) = \int F(y)\,dy$$

f, ϕ are two arbitrary functions, which is required solution of ①.

4 Solve
$$\frac{\partial^2 z}{\partial x^2} = a^2 z, \text{ when } x = 0, \frac{\partial z}{\partial x} = a \sin y \text{ and } \frac{\partial z}{\partial y}$$

Solution: The given equation is
$$\frac{\partial^2 z}{\partial x^2} = a^2 z, \text{ when } x = 0, \frac{\partial z}{\partial x} = a \sin y \text{ and } \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$$

$$\left(\frac{\partial^2}{\partial x^2} - a^2\right) z = 0$$

$\therefore$ Auxiliary equation is
$$m^2 - a^2 = 0 \quad , \quad \frac{\partial}{\partial x} = m,$$

$$m^2 = a^2$$

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$m = \pm a$

$\therefore$ $Z = c_1 e^{ax} + c_2 e^{-ax}$, if z is a function of x alone.

or $Z = A \sinh ax + B \cosh ax$ or

$A \cos ax + B \sinh ax$,

where A, B are constants.

Since z is a functions of x and y, A and B are the functions of y.

Then $z = \sinh ax \cdot f(y) + \cosh ax \cdot \phi(y)$. —— (1)

on putting $x = 0$ and $\frac{\partial z}{\partial x} = a \sin y$ in

$$\frac{\partial z}{\partial x} = \cosh ax \cdot a f(y) + \sinh ax \cdot a \phi(y)$$

Due to (1)

$$a \sin y = a \cdot f(y) \Rightarrow f(y) = \sin y$$

putting value of $F(y)$ in (1), we get

$$z = \sinh ax \cdot \sin y + \cosh ax \cdot \phi(y) \text{ — } (2)$$

Differentiating (2) with respect to y, we get

$$\frac{\partial z}{\partial y} = \sinh ax \cdot \cos y + \cosh ax \cdot \phi'(y) \text{ —— } (3)$$

on putting $x = 0$ and $\frac{\partial z}{\partial y} = 0$ in (3), we get

$$0 = \phi'(y)$$

Integrating with respect to y, we get

$$\phi(y) = \phi(x).$$

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Lagrange's Linear Equation :

Lagrange's linear equation is an equation of the type

$$Pp + Qq = R \,,$$

Where P, Q, R are the functions of x, y, z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

ition: Lagrange's linear equation is

$$Pp + Qq = R \,, \quad \text{——} \; (1)$$

where P, Q, R are functions of x, y, z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

This form of the equation is obtained by eliminating an arbitrary function f from

$$f(u, v) = 0 \quad \text{——} \; (2)$$

where u, v are functions of x, y, z.

Differentiating (2) partially with respect to x and y.

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0 \quad \text{and}$$

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0$$

$$\text{or} \quad \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) = -\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) \quad \text{——} \; (3)$$

$$\text{and} \quad \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) = -\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) \quad \text{——} \; (4)$$

Dividing (3) by (4) for eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, we get

$$\frac{\partial u/\partial x + p\,\partial u/\partial z}{\partial u/\partial y + q\,\partial u/\partial z} = \frac{\partial v/\partial x + p\,\partial v/\partial z}{\partial v/\partial y + q \cdot \partial v/\partial z}$$

$$\text{or} \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q \cdot \frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)$$

$$\text{or} \quad \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + q\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} + \underline{p\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} + \underline{pq\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}} =$$

$$\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \underline{p\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}} + q\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} + \underline{pq\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}}$$

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$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)P+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q$$
$$=\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \quad \text{——— (5)}$$

If (1) and (5) are the same, then the coefficients and q are equal.

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$$

$$Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$$

$$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Now suppose

$$u=u(x,y,z)=a \quad \text{——— (6)}$$

and

$$v=v(x,y,z)=b \quad \text{——— (7)}$$

are two solutions, where a, b are constants.

Differentiating (6) and (7), we get

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0 \quad \text{——— (8)} \quad \text{and}$$

$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0 \quad \text{——— (9)}$$

Solving (8) and (9), we get

$$\frac{dx}{\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\end{vmatrix}}=\frac{dy}{\begin{vmatrix}\frac{\partial u}{\partial z} & \frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x}\end{vmatrix}}=\frac{dz}{\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}}$$

$$\frac{dx}{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Solution of these equations are

$u=u(x,y,z)=a$ and $v=v(x,y,z)=b$

$\therefore f(u,v)=0$ is the required solution of (1).

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Working Rule:

Given equation is $Pp + Qq = R$

First step:

Write down the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Second step:

Solve the above auxiliary equations. Let the two solutions be
$u = u(x, y, z) = c_1$ and $v = v(x, y, z) = c_2$,
where c_1 and c_2 are constants.

Third step:

Then $F(u, v) = 0$ or $f(u, v) = 0$ or
$u = \phi(v)$ or $v = \psi(u)$ is the required solution of

$$Pp + Qq = R.$$

Solve the following partial differential equation

$yq = xp + z$ or $yq - xp = z$, where $p = \frac{\partial z}{\partial x}$,

$q = \frac{\partial z}{\partial y}$.

Solution: The given partial differential equation is

$$yq - xp = z \quad \text{or} \quad y\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial x} = z$$

$$\text{or} \quad -x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

which is of the type

$$Pp + Qq = R$$

$\therefore$ It's auxiliary equation are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$

consider first two ratio

$$\frac{dx}{-x} = \frac{dy}{y}$$

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Integrating, we get

$$-\log x + \log a = \log y.$$
$$-\log x - \log y = -\log a.$$
$$\log x + \log y = \log a$$
$$\log xy = \log a$$
$$\therefore xy = a = u = u(x,y,z)$$

Now consider last two ratio

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get

$$\log y = \log z + \log b$$
$$\log y - \log z = \log b$$
$$\log y/z = \log b$$
$$\therefore y/z = b = v = v(x,y,z)$$

Hence the solution is

$$f(xy, y/z) = 0$$

where f is an arbitrary function.

Ex 7 Solve.

$$y^2 p - xyq = x(z-2y)$$

solution: The given equation is

$$y^2 p - xyq = x(z-2y)$$
$$y^2 \frac{\partial z}{\partial x} - xy \frac{\partial z}{\partial y} = x(z-2y) \quad \text{——— (1)}$$

which is of the type

$$Pp + Qq = R$$

∴ It's auxiliary equation are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$
$$\therefore \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

consider first two ratio.

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$
$$\frac{dx}{y} = \frac{dy}{-x}$$

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$$-x\,dx = y\,dy$$
$$x\,dx + y\,dy = 0$$

Integrating the above equation, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = \text{constant}$$

$$x^2 + y^2 = \text{constant} = c_1, \text{ say}$$

Now consider last two ratio.

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dy}{-y} = \frac{dz}{z-2y}$$

or $\quad z\,dy - 2y\,dy = -y\,dz$

or $\quad y\,dz + z\,dy = 2y\,dy$

or $\quad d(yz) = 2y\,dy$.

Integrating, we get

$$y^2 = yz + c_2$$
$$y^2 - yz = c_2$$

Hence the solution is

$$f(x^2+y^2,\ y^2-yz) = 0$$

or $\quad x^2+y^2 = \phi(y^2-yz)$

where ϕ is an arbitrary function.

* Solve

$$(x^2-yz)p + (y^2-zx)q = z^2-xy.$$

lution: The given equation is

$$(x^2-yz)p + (y^2-zx)q = z^2-xy. \quad \text{—} (1)$$

$$(x^2-yz)\frac{\partial z}{\partial x} + (y^2-zx)\frac{\partial z}{\partial y} = z^2-xy.$$

which is of the type

$$Pp + Qq = R$$

$\therefore$ Its auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

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or $\dfrac{dx-dy}{x^2-yz-(y^2-zx)} = \dfrac{dy-dz}{y^2-zx-(z^2-xy)} = \dfrac{dz-dx}{z^2-xy-(x\ldots}$

or $\dfrac{dx-dy}{(x^2-y^2)-yz+zx} = \dfrac{dy-dz}{(y^2-z^2)-zx+xy} = \dfrac{dz-dx}{(z^2-x^2)-xy+\ldots}$

or $\dfrac{dx-dy}{(x-y)(x+y)+z(x-y)} = \dfrac{dy-dz}{(y-z)(y+z)+x(y-z)}$

$= \dfrac{dz-dx}{(z-x)(z+x)+y(z-x)}$

or $\dfrac{dx-dy}{(x-y)(x+y+z)} = \dfrac{dy-dz}{(y-z)(x+y+z)} = \dfrac{dz-dx}{(z-x)(x+y+z)}$

or $\dfrac{dx-dy}{x-y} = \dfrac{dy-dz}{y-z} = \dfrac{dz-dx}{z-x}$ —— (2)

consider first two ratio of (2)

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

Integrating, we get

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log(x-y) - \log(y-z) = \log c_1$$

$$\log\frac{(x-y)}{(y-z)} = \log c_1$$

$$\therefore \frac{x-y}{y-z} = c_1$$

Now consider last two ratio of (2)

$$\frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

Integrating, we get

$$\log(y-z) = \log(z-x) + \log c_2$$

$$\log(y-z) - \log(z-x) = \log c_2$$

$$\log\frac{(y-z)}{(z-x)} = \log c_2$$

$$\therefore \frac{y-z}{z-x} = c_2$$

$\therefore$ The required solution of (1) is

$$f\left[\frac{x-y}{y-z}, \frac{y-z}{z-x}\right] = 0$$

where f is an arbitrary function.

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Method of Multipliers :

Let the auxiliary equation be $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ of $Pp + Qq = R$.

l, m, n may be constants or functions of x, y, z, then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

l, m, n are choosen in such a way that

$$lP + mQ + nR = 0$$

Thus $\quad l\,dx + m\,dy + n\,dz = 0$

Solve this differential equation, if the solution is

$$u = u(x, y, z) = c_1$$

similarly, choose another set of multipliers l_1, m_1, n_1.

That is, $\quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l_1 dx + m_1 dy + n_1 dz}{l_1 P + m_1 Q + n_1 R}$

l_1, m_1, n_1 are choosen in such a way that

$$l_1 P + m_1 Q + n_1 R = 0$$

Thus $\quad l_1 dx + m_1 dy + n_1 dz = 0$

Solve this differential equation, if the solution is

$$v = v(x, y, z) = c_2$$

$\therefore$ Required solution is $f(u, v) = 0$.

3. Solve

$$(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$$

Solution: The given equation is

IMP $\quad (mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx \quad \text{———} (1)$

$\therefore$ The auxiliary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using multipliers x, y, z, we get

Each fraction $= \frac{x\,dx + y\,dy + z\,dz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$

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$$= \frac{xdx + ydy + zdz}{xmz - xny + ynx - ylz + zly - zmx}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$\therefore xdx + ydy + zdz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \text{constant}.$$

$$x^2 + y^2 + z^2 = \text{constant} = c_1, \text{ say}$$

Again, using multipliers l, m, n, we get

$$\text{Each fraction} = \frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)}$$

$$= \frac{ldx + mdy + ndz}{lmz - lny + mnx - mlz + nly - nmx}$$

$$= \frac{ldx + mdy + ndz}{0}$$

$\therefore ldx + mdy + ndz = 0$

Integrating, we get

$lx + my + nz = c_2$

$\therefore$ The required solution of (1) is

$F(x^2 + y^2 + z^2, lx + my + nz) = 0$

or $x^2 + y^2 + z^2 = f(lx + my + nz)$.

Ex. 10. Find the general solution of

$$x(z^2 - y^2)\frac{\partial z}{\partial x} + y(x^2 - z^2)\frac{\partial z}{\partial y} = z(y^2 - x^2)$$

Solution: The given equation is

$$x(z^2 - y^2)\frac{\partial z}{\partial x} + y(x^2 - z^2)\frac{\partial z}{\partial y} = z(y^2 - x^2). \quad \text{——— (1)}$$

$\therefore$ The auxiliary equations are

$$\frac{\partial x}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \quad \text{——— (2)}$$

Using multipliers x, y, z, we get

Each term of (2) is equal to

$$\frac{xdx + ydy + zdz}{x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2z^2 - x^2y^2 + y^2x^2 - y^2z^2 + z^2y^2 - z^2x^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore \; x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \text{constant}$$

$$x^2 + y^2 + z^2 = \text{constant} = c_1 \text{, say}$$

Again, using multipliers $1/x$, $1/y$, $1/z$, we get

Each term of ② is equal to

$$\frac{(1/x)\,dx + (1/y)\,dy + (1/z)\,dz}{(z^2-y^2) + (x^2-z^2) + (y^2-x^2)}$$

$$= \frac{(1/x)\,dx + (1/y)\,dy + (1/z)\,dz}{0}$$

$$\therefore \; \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2$$

$$\therefore \; xyz = c_2$$

$\therefore$ The required general solution is

$$F(x^2+y^2+z^2,\, xyz) = 0$$

or $\quad xyz = f(x^2+y^2+z^2).$

Q. 11. Solve the partial differential equation

$$\frac{y-z}{yz}\,p + \frac{z-x}{zx}\,q = \frac{x-y}{xy}$$

Solution: The given partial differential equation is

$$\frac{y-z}{yz}\,p + \frac{z-x}{zx}\,q = \frac{x-y}{xy}$$

Multiplying by xyz, we get

$$x(y-z)p + y(z-x)q = z(x-y)$$

$\therefore$ The auxiliary equations are

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$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$\therefore$ Each ratio $= \dfrac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)}$

$= \dfrac{dx+dy+dz}{xy-xz+yz-yx+zx-zy}$

$= \dfrac{dx+dy+dz}{0}$

$\therefore dx+dy+dz=0$

Integrating, we get

$x+y+z=a$

Again each ratio $= \dfrac{(1/x)dx+(1/y)dy+(1/z)dz}{y-z+z-x+x-y}$.

$= \dfrac{(1/x)dx+(1/y)dy+(1/z)dz}{0}$

$\therefore \dfrac{dx}{x}+\dfrac{dy}{y}+\dfrac{dz}{z}=0$

Integrating, we get

$\log x+\log y+\log z=\log b$.

$\log xyz = \log b$

$xyz = b$

$\therefore$ The required general solution is

$F(x+y+z,\ xyz)=0$

or $xyz = f(x+y+z)$.

Ex. 12 Solve

$$(x^2-y^2-z^2)p+2xyq = 2xz$$

Solution: The given equation is

$$(x^2-y^2-z^2)p+2xyq = 2xz \quad \text{—— (1)}$$

$\therefore$ The auxiliarly equations are

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad \text{—— (2)}$$

consider last two ratio

$$\frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

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Integrating, we get

$$\log y = \log z + \log a$$

$$\log y = \log z + \log a$$

$$\log y - \log z = \log a$$

$$\log y/z = \log a$$

$$\therefore \; y/z = a$$

Using multipliers x, y, z, we have

Each term of ② is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2-y^2-z^2) + 2xy^2 + 2xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(x^2-y^2-z^2+2y^2+2z^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

Now consider this ratio with last ratio

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)} = \frac{dz}{2xz}$$

$$\therefore \; \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} = \frac{dz}{z}$$

Integrating, we get

$$\log(x^2+y^2+z^2) - \log z = \log b$$

$$\log\left(\frac{x^2+y^2+z^2}{z}\right) = \log b$$

$$\therefore \; \frac{x^2+y^2+z^2}{z} = b$$

$\therefore$ The required general solution of ① is

$$F\left(y/z,\ \frac{x^2+y^2+z^2}{z}\right) = 0$$

$$\text{or} \quad \frac{x^2+y^2+z^2}{z} = f(y/z)$$

$$\text{or} \quad x^2+y^2+z^2 = z f(y/z)$$

3. Solve the differential equation

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z$$

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Solution: The given equation is

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z \quad \text{——} (1)$$

The auxiliary equations of (1) are

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{z(x+y)} \quad \text{——} (2)$$

Take first two member of (2)

$$\frac{dx}{x^2}=\frac{dy}{y^2}$$

$$x^{-2}dx=y^{-2}dy$$

Integrating, we get

$$\frac{x^{-1}}{-1}=\frac{y^{-1}}{-1}+\text{constant}.$$

$$-\frac{1}{x}=-\frac{1}{y}+\text{constant}.$$

$$\frac{1}{x}=\frac{1}{y}-\text{constant}$$

$$\frac{1}{x}-\frac{1}{y}=c_1$$

$$\text{Each ratio}=\frac{(1/x)dx+(1/y)dy+(-1/z)dz}{x+y-(x+y)}$$

$$=\frac{(1/x)dx+(1/y)dy-(1/z)dz}{0}$$

$$\therefore \frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}=0$$

Integrating, we get

$$\log x+\log y-\log z=\log c_2$$

$$\log xy-\log z=\log c_2$$

$$\log\frac{xy}{z}=\log c_2$$

$$\therefore xy/z=c_2$$

$\therefore$ The required general solution of (1) is

$$F\left(\frac{1}{x}-\frac{1}{y},\frac{xy}{z}\right)=0$$

or $\left(\frac{1}{x}-\frac{1}{y}\right)=f\left(\frac{xy}{z}\right)$

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Find the general solution of

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt$$

Solution: The given equation is

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt \quad \text{---} (1)$$

$\therefore$ The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{xyt} \quad \text{---} (2)$$

Take first two members of (2)

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

$\log x = \log y + \log a$

$\log x = \log ya$

$\therefore x = ya$

or $\frac{x}{y} = a$ or $\frac{y}{x} = a_1$, where $a_1 = 1/a$

Take second and third members of (2)

$$\frac{dy}{y} = \frac{dt}{t}$$

Integrating, we get

$\log y = \log t + \log b$

$\log y - \log t = \log b$

$\log y/t = \log b$ or $t/y = 1/b = b_1$, say

$\therefore y/t = b$

multiplying the equation (2) by xyt, we get

$$dz = \frac{ytdx}{1} = \frac{txdy}{1} = \frac{xydt}{1} = \frac{ytdx + txdy + xydt}{3}$$

$$dz = \frac{1}{3}d(xyt)$$

Integrating, we get

$$z = \frac{1}{3}xyt + c_1$$

$$z - \frac{1}{3}xyt = c_1$$

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$\therefore$ The required general solution of ① is

$$F\left(y/x,\ t/y,\ z-\frac{1}{3}xyt\right)=0.$$

or $z-\frac{1}{3}xyt = f\left(y/x,\ t/y\right)$.

Ex.15. Solve

$$(y+z)p-(x+z)q=x-y$$

Solution: The given equation is

$$(y+z)p-(x+z)q=x-y \quad \text{——— ①}$$

$\therefore$ The auxiliary equation are

$$\frac{dx}{y+z}=\frac{dy}{-(x+z)}=\frac{dz}{(x-y)}$$

Each ratio $= \dfrac{dx+dy+dz}{y+z-x-z+x-y}$

$= \dfrac{dx+dy+dz}{0}$

$\therefore dx+dy+dz=0$

Integrating, we get

$x+y+z=c_1$

Now each ratio $= \dfrac{xdx+ydy+(-z)dz}{x(y+z)-y(x+z)-z(x-y)}$

$= \dfrac{xdx+ydy-zdz}{xy+xz-yx-yz-zx+zy}$

$= \dfrac{xdx+ydy-zdz}{0}$

$\therefore xdx+ydy-zdz=0$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2}=\text{constant}.$$

$x^2+y^2-z^2=\text{constant}=c_2$, say

$\therefore$ The required general solution ① is

$$f(x+y+z,\ x^2+y^2-z^2)=0$$

Ex-16 Solve

$$zp+yq=x.$$

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Solution: The given equation is

$$zp + yq = x \quad \text{——} (1)$$

The auxiliary equations are

$$\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x} \quad \text{——} (2)$$

consider first and third ratio of (2)

$$\frac{dx}{z} = \frac{dz}{x}$$

$$xdx = zdz$$

Integrating, we get

$$\frac{z^2}{2} = \frac{x^2}{2} + \text{constant } (= c_1)$$

$$z = \sqrt{x^2 + c_1}$$

putting the value of z in (2) and consider first two ratio.

$$\frac{dx}{\sqrt{x^2+c_1}} = \frac{dy}{y}$$

Integrating, we get

$$\sin^{-1}\left(\frac{x}{\sqrt{c_1}}\right) = \log y + c_2 \qquad \because \int dx/\sqrt{x^2+(\sqrt{c_1})^2} = \sin^{-1}\frac{x}{\sqrt{c_1}}$$

$$\sin^{-1}\left(\frac{x}{\sqrt{c_1}}\right) - \log y = c_2$$

$\therefore$ The required general solution is

$$F\left((z^2 - x^2), \sin^{-1}\frac{x}{\sqrt{c_1}} - \log y\right) = 0$$

or $\sin^{-1}\frac{x}{\sqrt{c_1}} - \log y = f(z^2 - x^2)$

Ex. 16 Solve

$$px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$$

Solution: The given equation is

$$x(z-2y^2)p = (z-qy)(z-y^2-2x^3)$$

$$x(z-2y^2)p + y(z-y^2-2x^3)q = z(z-y^2-2x^3) \quad \text{——} (1)$$

$\therefore$ The auxiliary equations are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \quad \text{——} (2)$$

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consider the last two members of (2)

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get

$$\log y = \log z + \log a$$
$$\log y - \log z = \log a$$
$$\log y/z = \log a$$
$$\therefore y/z = a \quad \text{or } y = az \text{ ——— (3)}$$

Now consider the first and third members of (2)

$$\frac{dx}{x(z-2y^2)} = \frac{dz}{z(z-y^2-2x^3)}$$

By (3)

or. $$\frac{dx}{x(z-2a^2z^2)} = \frac{dz}{z(z-a^2z^2-2x^3)}$$

or $$\frac{dx}{xz(1-2a^2z)} = \frac{dz}{z(z-a^2z^2-2x^3)}$$

or $$\frac{dx}{x(1-2a^2z)} = \frac{dz}{z-a^2z^2-2x^3}$$

or $$(z-a^2z^2-2x^3)\,dx = x(1-2a^2z)\,dz$$

or $$z\,dx - a^2z^2dx - 2x^3dx = x\,dz - 2a^2zx\,dz$$

or $$(x\,dz - z\,dx) - 2a^2zx\,dz + a^2z^2dx + 2x^3\,dx = 0$$

or $$(x\,dz - z\,dx) - a^2(2xz\,dz - z^2dx) + 2x^3dx = 0$$

or $$\frac{x\,dz - z\,dx}{x^2} - a^2\frac{2xz\,dz - z^2dx}{x^2} + 2x\,dx = 0$$

or $$d(z/x) - a^2d(z^2/x) + d(x^2) = 0$$

Integrating, we get

$$\frac{x}{z} - a^2\frac{z^2}{x} + x^2 = b$$

$\therefore$ The required general solution of (1) is

$$F\left(y/z, \frac{z}{x} - a^2\frac{z^2}{x} + x^2\right) = 0$$

or $$\frac{y}{z} = f\left(\frac{z}{x} - a^2\frac{z^2}{x} + x^2\right) \text{ ——— (4)}$$

or $$\frac{z}{x} - a^2\frac{z^2}{x} + x^2 = \phi(y/z)$$

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Exercise - 9.3.

Solve the following partial differential equations

1. $p\tan x + q\tan y = \tan z$

Solution: The given equation is

$$\tan x\, p + \tan y\, q = \tan z \quad \text{——} (1)$$

$\therefore$ The auxiliary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \quad \text{——} (2)$$

consider first two ratio.

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Integrating, we get

$$\int \cot x\, dx = \int \cot y\, dy + \log a$$

$$\log \sin x = \log \sin y + \log a$$

$$\log \sin x - \log \sin y = \log a$$

$$\log \frac{\sin x}{\sin y} = \log a$$

$$\therefore \frac{\sin x}{\sin y} = a$$

Now consider last two ratio

$$\frac{dy}{\tan y} = \frac{dz}{\tan z} \quad \text{or} \quad \cot y\, dy = \cot z\, dz$$

Integrating, we get

$$\log \sin y = \log \sin z + \log b$$

$$\log \sin y - \log \sin z = \log b$$

$$\log \frac{\sin y}{\sin z} = \log b$$

$$\therefore \frac{\sin y}{\sin z} = b$$

$\therefore$ The required general solution is

$$f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0.$$

2. $(y-z)p + (x-y)q = z-x$

Solution: The given equation is

$$(y-z)p + (x-y)q = z-x \quad \text{——} (1)$$

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$\therefore$ It's auxiliary equations are

$$\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x}$$

Each ratio $= \dfrac{dx+dy+dz}{(y-z)+(x-y)+(z-x)}$

$= \dfrac{dx+dy+dz}{0}$

$\therefore dx+dy+dz=0$

Integrating, we get

$x+y+z=c_1$

Now each ratio $= \dfrac{xdx+2zdy+2ydz}{x(y-z)+2z(x-y)+2y(z-x)}$

$= \dfrac{xdx+2zdy+2ydz}{xy-xz+2zx-2zy+2yz-2xy}$

$= \dfrac{xdx+2zdy+2ydz}{-xy+zx}$

$= \dfrac{xdx+2zdy+2ydz}{x(z-y)}$

$\therefore \dfrac{dx}{y-z} = \dfrac{xdx+2zdy+2ydz}{x(z-y)}$

$\therefore xdx+2zdy+2ydz+xdx=0$

$\therefore 2x\,dx+2d(yz)=0$

$\therefore d(x^2)+2d(yz)=0.$

Integrating, we get

$x^2+2yz=c_2$

$\therefore$ The required general solution of (1) is

$f(x+y+z,\ x^2+2yz)=0.$

3. $(y+zx)p-(x+yz)q=x^2-y^2$

Solution: The given equation is

$(y+zx)p-(x+yz)q=x^2-y^2$ —— (1)

$\therefore$ The auxiliary equations are

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2}$$

Each ratio $= \dfrac{xdx+ydy-zdz}{x(y+zx)-y(x+yz)-z(x^2-y^2)}$

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$$= \frac{xdx + ydy - zdz}{xy + zx^2 - yx - zy^2 - zx^2 + zy^2}$$

$$= \frac{xdx + ydy - zdz}{0}$$

$\therefore \; xdx + ydy - zdz = 0.$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = \text{constant}.$$

$$x^2 + y^2 - z^2 = \text{constant} = c_1$$

Each ratio $= \dfrac{dx - dy}{y + zx + (x + yz)}$

$$= \frac{dx - dy}{(x+y) + z(x+y)}$$

$$= \frac{dx - dy}{(x+y)(z+1)}$$

$$\therefore \; \frac{dx - dy}{(x+y)(z+1)} = \frac{dz}{x^2 - y^2}.$$

$$\therefore \; \frac{dx - dy}{(x+y)(z+1)} = \frac{dz}{(x-y)(x+y)}.$$

$$\therefore \; \frac{dx - dy}{z+1} = \frac{dz}{(x-y)}$$

$\therefore \; (x-y)(dx - dy) = (z+1)dz$

$\therefore \; (x-y)(dx - dy) - (z+1)dz = 0.$

Integrating, we get

$$\frac{(x-y)^2}{2} - \frac{(z+1)^2}{2} = \text{constant}.$$

$\therefore \; (x-y)^2 - (z+1)^2 = \text{constant} = c_2$

$\therefore$ The required general solution of (1) is

$$F\left(x^2 + y^2 - z^2, \; (x-y)^2 - (z+1)^2\right) = 0$$

or $f(x^2 + y^2 - z^2) = (x-y)^2 - (z+1)^2$

-1 $\quad zx \dfrac{\partial z}{\partial x} - zy \dfrac{\partial z}{\partial y} = y^2 - x^2.$

Solution: The given equation are,

$$zx \frac{\partial z}{\partial x} - zy \frac{\partial z}{\partial y} = y^2 - x^2. \quad \text{——— (1)}$$

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The auxiliary equations are

$$\frac{dx}{2x} = \frac{dy}{-2y} = \frac{dz}{y^2-x^2}$$

Each ratio $= \frac{xdx+ydy+zdz}{2x^2-2y^2+2y^2-2x^2}$

$= \frac{xdx+ydy+zdz}{0}$

$\therefore xdx+ydy+zdz=0$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = \text{constant}$$

$$x^2+y^2+z^2 = \text{constant} = c_1$$

consider first two ratio

$$\frac{dx}{x} = \frac{dy}{-y}$$

$$\frac{dx}{x}+\frac{dy}{y}=0$$

Integrating, we get

$\log x + \log y = \log c_2$

$\log xy = \log c_2$

$\therefore \quad xy = c_2$

$\therefore$ The required general solution of (1) is

$f(x^2+y^2+z^2, xy)=0.$

5. Solve

$$pz-qz = z^2+(x+y)^2$$

Solution: The given equation is

$$pz-qz = z^2+(x+y)^2 \quad \text{—— (1)}$$

$\therefore$ Its auxiliary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$$

consider first two ratio

$$\frac{dx}{z} = \frac{dy}{-z}$$

$\therefore \quad dx+dy=0.$

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Integrating, we get

$$x+y=c_1$$

Each ratio $= \dfrac{2(x+y)(dx+dy)+2zdz}{2(x+y).z-2(x+y).z+2z(z^2+(x+y)^2)}$

$$= \frac{2(x+y)(dx+dy)+2zdz}{2z(z^2+(x+y)^2)}$$

$$\therefore \frac{2(x+y)(dx+dx)+2zdz}{2z(z^2+(x+y)^2)} = \frac{dx}{z}$$

$$\therefore \frac{2(x+y)(dx+dy)+2zdz}{z^2+(x+y)^2} = 2dx$$

Integrating, we get

$$\log[z^2+(x+y)^2] = 2x+\log c_2$$

$$\log[z^2+(x+y)^2]-\log c_2 = 2x$$

$$\log\frac{[z^2+(x+y)^2]}{c_2} = 2x$$

$$\therefore \frac{z^2+(x+y)^2}{c_2} = e^{2x}$$

$$\therefore [z^2+(x+y)^2]\, e^{-2x} = c_2$$

$\therefore$ The required general solution of ① is

$$F\left(x+y,\ (z^2+(x+y)^2)\, e^{-2x}\right)=0.$$

or $\quad (z^2+(x+y)^2)\, e^{-2x} = f(x+y).$

Solve

$$p+q+2xz=0.$$

Solution: The given equation is

$$p+q+2xz=0. \qquad \text{①}$$

$$p+q=-2xz$$

$\therefore$ It's auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{-2xz}$$

Consider first two ratio

$$dx=dy$$

$$dx-dy=0$$

Integrating, we get

$$x-y=c_1$$

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$$\text{Each ratio} = \frac{2x\,dx + (1/z)\,dz}{2x + (1/z)(-2xz)}$$

$$= \frac{2x\,dx + (1/z)\,dz}{2x - 2x}$$

$$= \frac{2x\,dx + (1/z)\,dz}{0}$$

$\therefore\ 2x\,dx + (1/z)\,dz = 0$

Integrating, we get

$x^2 + \log z = c_2$

$\therefore$ The required general solution is

$f(x-y) = x^2 + \log z$

7. Solve

$$x^2p + y^2q + z^2 = 0$$

Solution: The given equation is

$$x^2p + y^2q + z^2 = 0$$

$$x^2p + y^2q = -z^2 \quad \text{——— (1)}$$

$\therefore$ It's auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

consider first two ratio

$$x^{-2}dx = y^{-2}dy.$$

Integrating, we get

$$\frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + \text{constant}.$$

$$-\frac{1}{x} + \frac{1}{y} = \text{constant} = c_1$$

Now consider last two ratio

$$y^{-2}dy = -z^{-2}dz$$

Integrating, we get

$$\frac{y^{-1}}{-1} = z^{-1} + \text{constant}$$

$$\frac{-1}{y} - \frac{1}{z} = \text{constant}$$

$$\frac{1}{y} + \frac{1}{z} = c_2$$

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$\therefore$ The required general solution of (1) is

$$f\left(\frac{1}{y}-\frac{1}{x}, \frac{1}{y}+\frac{1}{z}\right)=0.$$

solve

$$(x^2+y^2)p+2xyq=(x+y)z.$$

The given equation is

$$(x^2+y^2)p+2xyq=(x+y)z \quad \text{—(1)}$$

$\therefore$ It's auxiliary equations are

$$\frac{dx}{x^2+y^2}=\frac{dy}{2xy}=\frac{dz}{(x+y)z}$$

Each ratio $= \dfrac{dx+dy}{x^2+y^2+2xy}$

$$=\frac{dx+dy}{(x+y)^2}$$

$$\therefore \frac{dx+dy}{(x+y)^2}=\frac{dz}{(x+y)z}$$

$$\therefore \frac{dx+dy}{x+y}-\frac{dz}{z}=0$$

Integrating, we get

$$\log(x+y)-\log z=\log c_1$$

$$\log\left(\frac{x+y}{z}\right)=\log c_1$$

$$\therefore \frac{x+y}{z}=c_1$$

Now each ratio $= \dfrac{x\,dx-y\,dy}{x(x^2+y^2)-y(2xy)}$

$$=\frac{x\,dx-y\,dy}{x^3+xy^2-2xy^2}$$

$$=\frac{x\,dx-y\,dy}{x^3-xy^2}$$

$$=\frac{x\,dx-y\,dy}{x(x^2-y^2)}$$

$$=\frac{2x\,dx-2y\,dy}{2x(x^2-y^2)}$$

$$\therefore \frac{dy}{2xy}=\frac{2x\,dx-2y\,dy}{2x(x^2-y^2)}$$

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$$\therefore \frac{dy}{y} = \frac{2x\,dx - 2y\,dy}{x^2-y^2}$$

Integrating, we get

$$\log y = \log(x^2-y^2) + \log c$$

$$\log y - \log(x^2-y^2) = \log c$$

$$\log \frac{y}{x^2-y^2} = \log c$$

$$\therefore \frac{y}{x^2-y^2} = c$$

$$\therefore \frac{2y}{x^2-y^2} = 2c = c_2, \text{ say}$$

$\therefore$ The required general solution of (1) is

$$f\left(\frac{x+y}{z}, \frac{2y}{x^2-y^2}\right) = 0$$

9. solve

$$\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} = 2x - e^y + 1.$$

Solution: The given equation is

$$\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} = 2x - e^y + 1 \quad \text{—— (1)}$$

$\therefore$ It's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{2x-e^y+1} \quad \text{—— (2)}$$

consider first two ratio

$$\frac{dx}{1} = \frac{dy}{-2}$$

$$-2dx = dy$$

$$2dx + dy = 0$$

Integrating, we get

$$2x + y = c_1$$

Each ratio $= \dfrac{-\left(\frac{2x+1}{2}\right)dx - \frac{e^y}{2}dy + dz}{-\left(\frac{2x+1}{2}\right)\cdot 1 - \frac{e^y}{2}x - 2 + 2x - e^y + 1}$

$$= \frac{dz - \left(\frac{2x+1}{2}\right)dx - \frac{e^y}{2}dy}{-\left(\frac{2x+1}{2}\right) + (2x+1)}$$

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$$= \frac{dz - \left(\frac{2x+1}{2}\right)dx - \frac{e^y}{2}dy}{\frac{2x+1}{2}}$$

$$\therefore \frac{dz - \left(\frac{2x+1}{2}\right)dx - \frac{e^y}{2}dy}{\frac{2x+1}{2}} = dx$$

$$\therefore dz - \left(\frac{2x+1}{2}\right)dx - \frac{e^y}{2}dy - \left(\frac{2x+1}{2}\right)dx = 0 .$$

$$\therefore dz - (2x+1)dx - \frac{e^y}{2}dy = 0$$

Integrating, we get

$$z - \frac{(2x+1)^2}{4} - \frac{e^y}{2} = c_2, \quad \text{where } t = 2x+1$$

$$dt = 2dx$$

$$dt/2 = dx$$

$\therefore$ The required general solution of (1) is

$$f(2x+y) = z - \frac{(2x+1)^2}{4} - \frac{e^y}{2}$$

Solve

$$p + 3q = 5z + \tan(y - 3x)$$

Sol^n: The given equation is

$$p + 3q = 5z + \tan(y-3x) \text{ ——— (1)}$$

$\therefore$ It's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$$

consider first two ratio

$$dx = \frac{dy}{3}$$

$$dy - 3dx = 0$$

Integrating, we get

$$y - 3x = c_1$$

Now consider first and third ratio.

$$dx = \frac{dz}{5z + \tan(y-3x)} \qquad \because c_1 = y - 3x$$

$$5dx = \frac{5dz}{5z + \tan c_1}$$

Integrating, we get.

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$\log(5z+\tan c_1)+\log c_2 = 5x$, where $t = 5z+\tan c_1$

$\log(5z+\tan(y-3x)) \cdot c_2 = 5x$

$$5z+\tan(y-3x) \cdot c_2 = e^{5x}$$

$$c_2 = \frac{e^{5x}}{5z+\tan(y-3x)}$$

$\therefore$ The required general solution of ① is

$$f(y-3x) = \frac{e^{5x}}{5z+\tan(y-3x)}$$

11 Solve

$$xp - yq + x^2 - y^2 = 0$$

Solution: The given equation is

$$xp - yq + x^2 - y^2 = 0$$

$$xp - yq = y^2 - x^2 \text{ ——— ①}$$

$\therefore$ Its auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2-x^2}$$

consider first two ratio

$$\frac{dx}{x} = \frac{dy}{-y}$$

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating, we get

$$\log x + \log y = \log c_1$$

$$\log xy = \log c_1$$

$$\therefore xy = c_1$$

Each ratio $= \dfrac{xdx + ydy + dz}{x^2-y^2+y^2-x^2}$

$$= \frac{xdx+ydy+dz}{0}$$

$\therefore xdx + ydy + dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + z = c_2$$

$\therefore$ The required general solution of ① is

$$f(xy) = \frac{x^2}{2} + \frac{y^2}{2} + z.$$

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solve

$$(x+y)\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)=z-1$$

ution: The given equation is

$$(x+y)\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)=z-1$$

$$(x+y)p+(x+y)q=z-1 \quad \text{—— (1)}$$

$\therefore$ Its auxiliary equations are

$$\frac{dx}{x+y}=\frac{dy}{x+y}=\frac{dz}{z-1}$$

consider first two ratio

$$\frac{dx}{x+y}=\frac{dy}{x+y}$$

$$dx-dy=0$$

Integrating, we get

$$x-y=c_1$$

Each ratio $=\dfrac{dx+dy}{2(x+y)}$

$$\therefore \frac{dx+dy}{2(x+y)}=\frac{dz}{z-1}$$

$$\therefore \frac{dx+dy}{x+y}=2\frac{dz}{z-1}$$

Integrating, we get

$$\log(x+y)=2\log(z-1)+\log c_2$$

$$\log(x+y)-2\log(z-1)=\log c_2$$

$$\log(x+y)-\log(z-1)^2=\log c_2$$

$$\log\frac{(x+y)}{(z-1)^2}=\log c_2$$

$$\therefore \frac{x+y}{(z-1)^2}=c_2$$

$\therefore$ The required general solution of (1) is

$$f(x-y)=\frac{x+y}{(z-1)^2}$$

13. Solve.

$$(x^3+3xy^2)\frac{\partial z}{\partial x}+(y^3+3x^2y)\frac{\partial z}{\partial y}=2(x^2+y^2)z$$

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Solution: The given equation is

$$(x^3+3xy^2)p+(y^3+3x^2y)q = 2(x^2+y^2)z \quad (1)$$

$\therefore$ Its auxiliary equations are

$$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2(x^2+y^2)z}$$

$$\text{Each ratio} = \frac{(1/x)dx+(1/y)dy-(2/z)dz}{(1/x)(x^3+3xy^2)+(1/y)(y^3+3x^2y)-(2/z)2(x^2+y^2)z}$$

$$= \frac{dx/x+dy/y-2dz/z}{x^2+3y^2+y^2+3x^2-4x^2-4y^2}$$

$$= \frac{dx}{x}+\frac{dy}{y}-\frac{2dz}{z} \Big/ 0$$

$$\therefore \frac{dx}{x}+\frac{dy}{y}-2\frac{dz}{z}=0.$$

Integrating, we get

$$\log x+\log y-2\log z = \log c_1$$

$$\log xy-\log z^2 = \log c_1$$

$$\log \frac{xy}{z^2} = \log c_1$$

$$\frac{xy}{z^2} = c_1$$

$$\text{Each ratio} = \frac{dx+dy}{x^3+3xy^2+y^3+3x^2y}$$

$$= \frac{dx+dy}{(x+y)^3}$$

$$\text{Each ratio} = \frac{dx-dy}{x^3+3xy^2-y^3-3x^2y}$$

$$= \frac{dx-dy}{(x-y)^3}$$

$$\therefore \frac{dx+dy}{(x+y)^3} = \frac{dx-dy}{(x-y)^3}$$

$$\therefore (x+y)^{-3}(dx+dy)-(x-y)^{-3}(dx-dy)=0.$$

Integrating, we get

$$\frac{(x+y)^{-2}}{-2}-\frac{(x-y)^{-2}}{-2} = \text{constant}.$$

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$$\frac{1}{(x+y)^2} - \frac{1}{(x-y)^2} = \text{constant} = c_2$$

∴ The required general solution is

$$f\left(\frac{xy}{z^2}\right) = \frac{1}{(x+y)^2} - \frac{1}{(x-y)^2}$$

solve

$$(z^2-2yz-y^2)p+(xy+zx)q = xy-zx.$$

Ans. The given equation is

$$(z^2-2yz-y^2)p+(xy+zx)q = xy-zx. \quad \text{—— (1)}$$

∴ It's auxiliary equations are

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx}$$

Each ratio $= \dfrac{xdx+ydy+zdz}{x(z^2-2yz-y^2)+y(xy+zx)+z(xy-zx)}$

$$= \frac{xdx+ydy+zdz}{xz^2-2xyz-xy^2+xy^2+xyz+xyz-xz^2}$$

$$= \frac{xdx+ydy+zdz}{0}$$

∴ $xdx+ydy+zdz = 0.$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = \text{constant}$$

$$x^2+y^2+z^2 = 2\text{constant} = c_1$$

consider last two ratio

$$\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

$$\frac{dy}{(y+z)} = \frac{dz}{y-z}$$

$$(y-z)dy-(y+z)dz = 0.$$

$$ydy-zdy-ydz-zdz = 0.$$

$$ydy-zdz-(ydz+zdy) = 0.$$

$$ydy-zdz-d(yz) = 0.$$

$$ydy-zdz-d$$

Integrating, we get.

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$$\frac{y^2}{2} - \frac{z^2}{2} - yz = \text{constant.}$$

$$y^2 - z^2 - 2yz = \text{constant} = C_2$$

$\therefore$ The required general solution of (1) is

$$f(x^2 + y^2 + z^2) = y^2 - z^2 - 2yz.$$

15. Find the solution of the equation

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0$$

which is passes through the curve $z = 1$, $x^2 + y^2 = 1$

Solution:

$$p^2 + q^2 = 1$$

$$z = ax + by = c$$

$$p = \frac{\partial z}{\partial x} = a \quad q = \frac{\partial z}{\partial y} = b$$

$$a^2 + b^2 = 1 \quad b = \sqrt{1 - a^2}$$

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PDE

UNIT - II

Art. 9.5 Partial Differential Equations Non-linear in p and q:

We give below the methods of solving non-linear partial differential equations in certain standard form only.

Type I: Equation of the type $f(p, q) = 0$
i.e. equations containing p and q only.

Method: Let the required solution be

$$z = ax + by + c$$

$$\therefore \frac{\partial z}{\partial x} = a, \quad \frac{\partial z}{\partial y} = b$$

$$\therefore \quad p = a, \quad q = b$$

On putting these values in $f(p, q) = 0$, we get

$$f(a, b) = 0$$

From this, find the value of b in terms of a and substitute the value of b in ①, that will be the required solution.

Ex. 18. Solve

$$p^2 + q^2 = 1$$

Solution: The given equation is

$$p^2 + q^2 = 1 \quad \text{— ①}$$

Let the required solution be

$$z = ax + by + c \quad \text{— ②}$$

Differentiating ② with respect to x and y, we get

$$\frac{\partial z}{\partial x} = p = a, \quad \frac{\partial z}{\partial y} = q = b$$

On putting the values of p and q in ①, we get

$$a^2 + b^2 = 1$$

$$\therefore \quad b^2 = 1 - a^2$$

$$\therefore \quad b = \sqrt{1 - a^2}$$

Put the value of b in ②, we get

$$z = ax + (\sqrt{1 - a^2})y + c$$

This is the required solution.

Ex. 19 Solve

$$x^2p^2 + y^2q^2 = z^2$$

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Solution: The given equation is

$$x^2p^2+y^2q^2=z^2$$

$$\frac{x^2}{z^2}p^2+\frac{y^2}{z^2}q^2=1$$

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2+\left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2=1$$

$$\left(\frac{\frac{\partial z}{z}}{\frac{\partial x}{x}}\right)^2+\left(\frac{\frac{\partial z}{z}}{\frac{\partial y}{y}}\right)^2=1 \quad \text{---} (1)$$

Let $\frac{\partial z}{z}=\partial Z$, $\frac{\partial x}{x}=\partial X$, $\frac{\partial y}{y}=\partial Y$

$\therefore \log z = Z$, $\log x = X$, $\log y = Y$

$\therefore$ Equation (1) can be written as

$$\left(\frac{\partial Z}{\partial X}\right)^2+\left(\frac{\partial Z}{\partial Y}\right)^2=1$$

or $\quad P^2+Q^2=1 \quad$ (2), $\quad P=\frac{\partial Z}{\partial X}$, $Q=\frac{\partial Z}{\partial Y}$

Let the required solution be

$$Z=aX+bY+c \quad \text{---} (3)$$

$\therefore \frac{\partial Z}{\partial X}=P=a$, $\frac{\partial Z}{\partial Y}=Q=b$

Putting the values of P and Q in (2), we get

$$a^2+b^2=1$$

$$\therefore \quad b^2=1-a^2$$

$$\therefore \quad b=\sqrt{1-a^2}$$

Put the value of b in (3), we get

$$Z=aX+(\sqrt{1-a^2})Y+c$$

$$\therefore \log z = a\log x+(\sqrt{1-a^2})\log y + c$$

Exercise 9.4

Solve the following partial differential equations

1. $pq=1$
2. $\sqrt{p}+\sqrt{q}=1$
3. $p^2-q^2=1$
4. $pq+p+q=0$

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Solution: The given equation is

(1) $$pq = 1 \quad \text{---} (1)$$

Let the required solution be

$$z = ax + by + c \quad \text{---} (2)$$

$$\therefore \frac{\partial z}{\partial x} = p = a, \quad \frac{\partial z}{\partial y} = q = b$$

On putting the values of p and q in (1), we get

$$ab = 1 \quad \text{or} \quad b = 1/a$$

Put the value of b in (2), we get

$$z = ax + (1/a)y + c$$

This is the required solution.

Solution: The given equation is

(2) $$\sqrt{p} + \sqrt{q} = 1 \quad \text{---} (1)$$

Let the required solution be

$$z = ax + by + c \quad \text{---} (2)$$

$$\therefore \frac{\partial z}{\partial x} = p = a, \quad \frac{\partial z}{\partial y} = q = b$$

On putting the values of p and q in (1), we get

$$\sqrt{a} + \sqrt{b} = 1$$

$$\text{or} \quad \sqrt{b} = 1 - \sqrt{a}$$

squaring from both sides, we get

$$b = (1 - \sqrt{a})^2$$

Put the value of b in (2), we get

$$z = ax + (1 - \sqrt{a})^2 y + c$$

This is the required solution.

Solution: The given equation is

(3) $$p^2 - q^2 = 1 \quad \text{---} (1)$$

Let the required solution be

$$z = ax + by + c \quad \text{---} (2)$$

$$\therefore \frac{\partial z}{\partial x} = p = a, \quad \frac{\partial z}{\partial y} = q = b$$

$$\therefore \quad p^2 = a^2, \quad q^2 = b^2$$

On putting the values of p^2 and q^2 in (1), we get

$$a^2 - b^2 = 1$$

$$b^2 = a^2 - 1$$

$$\therefore \quad b = \sqrt{a^2 - 1}$$

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Put the value of b in ②, we get

$z = ax + (\sqrt{a^2-1})y + c$

This is the required solution.

Solution: The given equation is

④ $pq + p + q = 0$ —— ①

Let the required solution be

$z = ax + by + c$ —— ②

$\therefore \frac{\partial z}{\partial x} = p = a, \quad \frac{\partial z}{\partial y} = q = b$

On substituting the values of p and q in ①, we get

$ab + a + b = 0$

$(a+1)b + a = 0$

$(a+1)b = -a$

$b = \frac{-a}{1+a}$

Put the value of b in ②, we get

$z = ax + \left(\frac{-a}{1+a}\right)y + c$

This is the required solution.

Type II: Equation of the type

$z = px + qy + f(p,q)$

Method: Let the required solution be

$z = ax + by + f(a,b)$,

where $p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = a$.

Ex. 20. Solve

$z = px + qy + p^2 + q^2$

Solution:- The given equation is

$z = px + qy + p^2 + q^2$ —— ①

$\therefore$ Replacing p by a and q by b, we get

$z = ax + by + a^2 + b^2$

Which is the required solution of ①

Ex. 21 Solve

$z = px + qy + 2\sqrt{pq}$

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Solution: The given equation is

$$z = px + qy + 2\sqrt{pq} \quad \text{——} \ (1)$$

∴ Replacing p by a and q by b, we get

$$z = ax + by + 2\sqrt{ab}$$

Which is the required solution of (1).

Type III: Equation of the type $f(z, p, q) = 0$
i.e. equations not containing x and y.

Imp

Method: Let z be a function of u where

$$u = x + ay$$

$$\therefore \frac{\partial u}{\partial x} = 1, \text{ and } \frac{\partial u}{\partial y} = a.$$

Then $p = \frac{\partial z}{\partial x}$

$$= \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$$

$$= \frac{dz}{du}$$

and $q = \frac{\partial z}{\partial y}$

$$= \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

$$= \frac{dz}{du}(a)$$

On putting the values of p and q in the given equation

$$f(z, p, q) = 0,$$

it becomes

$$f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$$

Which is an ordinary differential equation of the first order.

Rule: Assume $u = x + ay$, replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ in the given equation.

Then solve the ordinary differential equation, we get required solution.

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and a

Ex. 22. Solve

$$p(1+q) = qz$$

Solution: The given equation is

$$p(1+q) = qz \quad \text{——— (1)}$$

Let $u = x + ay \Rightarrow \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = a$, and z is a function of u.

Now $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

$= \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$ $\quad = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$

$= \frac{dz}{du}$ $\quad = \frac{dz}{du} . a$

On putting the values of p and q in (1), we get

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du} z$$

or $1 + a\frac{dz}{du} = az$

or $a\frac{dz}{du} = az - 1$

or $\frac{dz}{du} = \frac{az-1}{a}$

or $du = \frac{a\,dz}{az-1}$

Integrating, we get

$$u = \log(az-1) + \log c$$

$$x + ay = \log c(az-1)$$

Which is required solution of (1).

Ex. 23. Solve

$$p(1+q^2) = q(z-a)$$

Solution: The given equation is

$$p(1+q^2) = q(z-a) \quad \text{——— (1)}$$

Let z be a function of u where $u = x + by$.

$\therefore \frac{\partial u}{\partial x} = 1$ and $\frac{\partial u}{\partial y} = b$

Then $p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}(1) = \frac{dz}{du}$

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and $q = \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} = \frac{dz}{du}\cdot b = b\frac{dz}{du}$

On putting the values of p and q in ①, it becomes

$$\frac{dz}{du}\left(1+b^2\left(\frac{dz}{du}\right)^2\right) = b\frac{dz}{du}(z-a)$$

$$1+b^2\left(\frac{dz}{du}\right)^2 = b(z-a)$$

or $b^2\left(\frac{dz}{du}\right)^2 = bz-ab-1$

$$\left(\frac{dz}{du}\right)^2 = \frac{1}{b^2}(bz-ab-1)$$

$$\frac{dz}{du} = \frac{1}{b}\sqrt{bz-ab-1}$$

or $\frac{b\,dz}{\sqrt{bz-ab-1}} = du$

Integrating, we get

$2\sqrt{bz-ab-1} = u+c$, where $t = bz-ab-1$

$dt = b\,dz$

squaring from both sides.

$4.(bz-ab-1) = (u+c)^2$

$4(bz-ab-1) = (x+by+c)^2$

Which is the required solution of ①.

Ex. 24. Solve

$$z^2(p^2x^2+q^2) = 1.$$

Solution: The given equation is

$$z^2(p^2x^2+q^2) = 1 \quad \text{— ①}$$

Let z be a function of u where $u = X+ay$. — ②

$\because\; z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right] = 1$

or $z^2\left[\left(\frac{\partial z}{\frac{\partial x}{x}}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right] = 1$

or $z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right] = 1$, — ③

where $\frac{\partial x}{x} = \partial X$ or $\log x = X$ — ④

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From (2), we have

$\frac{\partial z}{\partial x} = p = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$ $\because \frac{\partial u}{\partial x} = 1$

$\frac{\partial z}{\partial y} = q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du} \cdot a$ $\because \frac{\partial u}{\partial y} = a$

On putting these values in (3), we get

$z^2 \left[\left(\frac{dz}{du}\right)^2 + \left(a \frac{dz}{du}\right)^2 \right] = 1$

or $\left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2 = \frac{1}{z^2}$

or $\left(\frac{dz}{du}\right)^2 (1+a^2) = \frac{1}{z^2}$

or $\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2(1+a^2)}$

or $\frac{dz}{du} = \frac{1}{z\sqrt{1+a^2}}$

or $z\, dz = \frac{du}{\sqrt{1+a^2}}$ or $(1+a^2)^{1/2} z\, dz = du$

Integrating, we get

$\sqrt{1+a^2} \cdot \frac{z^2}{2} = u + c$ or $\frac{z^2}{2} = \frac{u}{\sqrt{1+a^2}} + c$

$(\sqrt{1+a^2}) \cdot \frac{z^2}{2} = x + ay + c$ or $(\sqrt{1+a^2}) \frac{z^2}{2} = x + ay + c\sqrt{1+}$

$(\sqrt{1+a^2}) \frac{z^2}{2} = \log x + ay + c$ or $(\sqrt{1+a^2}) \frac{z^2}{2} = \log x + ay + c\sqrt{1+a}$

Which is required solution of (1).

Exercise 9.5

Solve

1. $z^2(p^2+q^2+1) = 1$.
2. $1+q^2 = q(z-a)$
3. $x^2p^2 + y^2q^2 = z$

Solution: The given equation is

(1) $z^2(p^2+q^2+1) = 1$ —— (1)

Let z be a function of u where $u = x+ay$.

$\therefore \frac{\partial u}{\partial x} = 1$, and $\frac{\partial u}{\partial y} = a$

Then $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

$= \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$ $\qquad = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$

$= \frac{dz}{du}$ $\qquad = a \frac{dz}{du}$

On putting these values of p and q in ①, we get

$$z^2 \left\{ \left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2 + 1 \right\} = 1$$

$$z^2 \left\{ (1+a^2) \left(\frac{dz}{du}\right)^2 \right\} + z^2 = 1$$

$$z^2 (1+a^2) \left(\frac{dz}{du}\right)^2 = 1 - z^2$$

$$z^2 \left(\frac{dz}{du}\right)^2 = \frac{1-z^2}{1+a^2}$$

$$\left(z \frac{dz}{du}\right) = \frac{\sqrt{1-z^2}}{\sqrt{1+a^2}}$$

$$\frac{z\,dz}{\sqrt{1-z^2}} = \frac{1}{\sqrt{1+a^2}} du$$

$$\frac{-z\,dz}{\sqrt{1-z^2}} = -\frac{1}{\sqrt{1+a^2}} du$$

Integrating, we get

$$\int \frac{(-z)\,dz}{\sqrt{1-z^2}} = -\frac{1}{\sqrt{1+a^2}} \int du + c$$

$$\frac{1}{2} \int \frac{dt}{t^{1/2}} = -\frac{1}{\sqrt{1+a^2}} \int du + c$$

Put $t = 1 - z^2$

$\therefore dt = -2z\,dz$

$\therefore \frac{dt}{2} = -z\,dz$

$$\frac{1}{2} \int t^{-1/2} dt = -\frac{1}{\sqrt{1+a^2}} \int du + c$$

$$\frac{1}{2} \frac{t^{1/2}}{1/2} = -\frac{1}{\sqrt{1+a^2}} u + c$$

$$(1-z^2)^{1/2} = -\frac{(x+ay)}{\sqrt{1+a^2}} + c$$

Which is the required solution of ①.

Solution: The given equation is

② $\qquad 1 + q^2 = q(z - a)$ $\qquad$ ①

Let z be a function of u where $u = x + by$

$\therefore \frac{\partial u}{\partial x} = 1$ and $\frac{\partial u}{\partial y} = b$

Then $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

$= \frac{dz}{du}\frac{\partial u}{\partial x}$; $= \frac{dz}{du}\frac{\partial u}{\partial y}$

$= \frac{dz}{du}.1$; $= \frac{dz}{du}.b$

$= \frac{dz}{du}$; $= b\frac{dz}{du}$

Putting these values of p and q in (1), we get

$$1 + b^2\left(\frac{dz}{du}\right)^2 = b\frac{dz}{du}(z-a)$$

$$b^2\left(\frac{dz}{du}\right)^2 - b(z-a)\frac{dz}{du} + 1 = 0$$

$$\left(\frac{dz}{du}\right)^2 - \frac{(z-a)}{b}\frac{dz}{du} + \frac{1}{b^2} = 0$$

This is a quadratic equation in $\frac{dz}{du}$

$$\therefore \frac{dz}{du} = \frac{(z-a)}{b} \pm \sqrt{\frac{(z-a)^2}{b^2} - 4\times1\times\frac{1}{b^2}} \Big/ 2\times1$$

$$= \frac{z-a}{b} \pm \frac{1}{b}\sqrt{(z-a)^2-4} \Big/ 2$$

$$= (z-a) \pm \sqrt{(z-a)^2-4} \Big/ 2b$$

$$\frac{du}{b} = \frac{2dz}{(z-a) \pm \sqrt{(z-a)^2-4}}$$

$$= \frac{2\left[(z-a) - \sqrt{(z-a)^2-4}\right]}{(z-a)^2 - (z-a)^2 + 4}dz$$

or

$$\frac{2\left[(z-a) + \sqrt{(z-a)^2-4}\right]}{(z-a)^2 - (z-a)^2 + 4}dz$$

By rationalizing

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$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + c$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$$

$$\frac{du}{b} = \frac{2[(z-a)-\sqrt{(z-a)^2-4}]}{4}dz \quad \text{or} \quad \frac{2[(z-a)+\sqrt{(z-a)^2-4}]}{4}dz$$

$$= \frac{1}{2}(z-a)\,dz - \frac{1}{2}\sqrt{(z-a)^2-2^2}\,dz$$

or

$$\frac{1}{2}(z-a)\,dz + \frac{1}{2}\sqrt{(z-a)^2-2^2}\,dz$$

Integrating, we get

$$\frac{u}{b} = \frac{(z-a)^2}{4} - \frac{1}{2}\left[\frac{z-a}{2}\sqrt{(z-a)^2-2^2} - \frac{2^2}{2}\log|z-a+\sqrt{(z-a)^2-2^2}|\right]$$

or

$$\frac{(z-a)^2}{4} + \frac{1}{2}\left[\frac{z-a}{2}\sqrt{(z-a)^2-2^2} - \frac{2^2}{2}\log|(z-a)+\sqrt{(z-a)^2-2^2}|\right]$$

$$\frac{x+by}{b} = \frac{(z-a)^2}{4} - \frac{(z-a)}{4}\sqrt{(z-a)^2-2^2} + \log|(z-a)+\sqrt{(z-a)^2-2^2}|$$

or

$$\frac{(z-a)^2}{4} + \frac{(z-a)}{4}\sqrt{(z-a)^2-2^2} - \log|(z-a)+\sqrt{(z-a)^2-2^2}|$$

OR

$$\frac{x+by}{b} = \frac{(z-a)^2}{4} - \frac{1}{2}\left[\frac{(z-a)}{2}\sqrt{(z-a)^2-2^2} - \frac{2^2}{2}\cosh^{-1}\left(\frac{z-a}{2}\right)\right]$$

or

$$\frac{(z-a)^2}{4} + \frac{1}{2}\left[\frac{z-a}{2}\sqrt{(z-a)^2-2^2} - \frac{2^2}{2}\cosh^{-1}\left(\frac{z-a}{2}\right)\right]$$

$$= \frac{(z-a)^2}{4} - \frac{1}{4}(z-a)\sqrt{(z-a)^2-2^2} + \cosh^{-1}\left(\frac{z-a}{2}\right)$$

or

$$\frac{(z-a)^2}{4} + \frac{1}{4}(z-a)\sqrt{(z-a)^2-2^2} - \cosh^{-1}\left(\frac{z-a}{2}\right)$$

Solution: The given equation is

(3) $\quad x^2p^2+y^2q^2 = z \quad$ — (1)

Putting $\frac{\partial x}{x} = \partial X$ i.e. $X = \log x$

and $\frac{\partial y}{y} = \partial Y$ i.e. $Y = \log y$

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$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial X}$ $\quad \because \frac{\partial X}{\partial x} = \frac{1}{x}$

$\therefore xp = \frac{\partial z}{\partial X} \Rightarrow (xp)^2 = \left(\frac{\partial z}{\partial X}\right)^2$

and $p = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial Y}$ $\quad \because \frac{\partial Y}{\partial y} = \frac{1}{y}$

$\therefore yq = \frac{\partial z}{\partial Y} \Rightarrow (yq)^2 = \left(\frac{\partial z}{\partial Y}\right)^2$

$\therefore$ The given equation ① reduces in the form

$$\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 = z$$

$$P^2 + Q^2 = z \quad \text{②} \quad , \quad P = \frac{\partial z}{\partial X}, \quad Q = \frac{\partial z}{\partial Y}$$

Which belongs to the standard form III, having X and Y as two independent variable, z being the dependent variable.

Hence, put $u = X + aY$

$$\therefore \frac{\partial u}{\partial X} = 1 \text{ and } \frac{\partial u}{\partial Y} = a$$

$$P = \frac{\partial z}{\partial X} = \frac{dz}{du}\frac{\partial u}{\partial X} = \frac{dz}{du} \cdot 1 = \frac{dz}{du}$$

$$Q = \frac{\partial z}{\partial Y} = \frac{dz}{du}\frac{\partial u}{\partial Y} = \frac{dz}{du}\, a$$

Putting these values in ②, we get

$$\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z$$

$$(1+a^2)\left(\frac{dz}{du}\right)^2 = z$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z}{a^2+1}$$

$$\therefore \frac{dz}{du} = \frac{z^{1/2}}{\sqrt{1+a^2}}$$

$$\therefore \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}}\, du$$

Integrating, we get

$$2\sqrt{z} = \frac{u}{\sqrt{1+a^2}} + c = \frac{X + aY}{\sqrt{1+a^2}} + c$$

$$\therefore 2\sqrt{z} = \frac{\log x + a\log y}{\sqrt{\quad}} + c$$

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Type IV :- Equation of the type
$$f_1(x,p) = f_2(y,q).$$
The given equation is of type
$$f_1(x,p) = f_2(y,q)$$
In these equations, z is absent and the terms containing x and p can be written on one side and the terms containing y and q can be written on the other side.

Method.

Let $f_1(x,p) = f_2(y,q) = a$

$f_1(x,p) = a$, solve it for p. Let $p = F_1(x)$

$f_2(y,q) = a$, solve it for q. Let $q = F_2(y)$

Since $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad \because z = z(x,y)$
$$= p\,dx + q\,dy$$
$$= F_1(x)\,dx + F_2(y)\,dy$$
Integrating, we get
$$z = \int F_1(x)\,dx + \int F_2(y)\,dy + c$$

Ex. 25. Solve $p - x^2 = q + y^2$ Imp

Sol The given equation is
$$p - x^2 = q + y^2 = a \text{, say}$$
$\therefore p - x^2 = a$ and $q + y^2 = a$

$\therefore p = x^2 + a$ and $q = a - y^2$

Putting these values of p and q in
$$dz = p\,dx + q\,dy$$
$$= (x^2+a)\,dx + (a-y^2)\,dy$$
Integrating, we get
$$z = \frac{x^3}{3} + ax + ay - \frac{y^3}{3} + a_1$$

Ex. 26. Solve $p^2 + q^2 = z^2(x+y)$

Solution The given equation is
$$p^2 + q^2 = z^2(x+y)$$

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$$\left(\frac{p}{z}\right)^2+\left(\frac{q}{z}\right)^2 = x+y$$

or $\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)^2+\left(\frac{1}{z}\frac{\partial z}{\partial y}\right)^2 = x+y$

or $\left(\frac{\partial z/z}{\partial x}\right)^2+\left(\frac{\partial z/z}{\partial y}\right)^2 = x+y$

or $\left(\frac{\partial z_1}{\partial x}\right)^2+\left(\frac{\partial z_1}{\partial y}\right)^2 = x+y$, where $\frac{\partial z}{z} = \partial z_1$

or $\log z = z_1$

or $P^2+Q^2 = x+y$

where $P = \frac{\partial z_1}{\partial x}$, $Q = \frac{\partial z_1}{\partial y}$

or $P^2 - x = y - Q^2 = a$, say

$\therefore P^2 - x = a$ or $y - Q^2 = a$

$\therefore P^2 = x+a$ or $y-a = Q^2$

$\therefore P = \sqrt{x+a}$ or $Q = \sqrt{y-a}$

Putting these values of P and Q in

$dz_1 = \frac{\partial z_1}{\partial x}dx + \frac{\partial z_1}{\partial y}dy$

$= P\,dx + Q\,dy$

$= \sqrt{x+a}\,dx + \sqrt{y-a}\,dy$

Integrating, we get

$z_1 = \frac{(a+x)^{3/2}}{3/2} + \frac{(y-a)^{3/2}}{3/2} + c$

$\log z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c$

Exercise 9.6

Solve:

1. $q - p + x - y$
2. $\sqrt{p} + \sqrt{q} = 2x$
3. $z = xp + p^2$
4. $z^2(p^2+q^2) = x^2+y^2$

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5. $z(p^2 - q^2) = x - y$.

6. $p^2 - q^2 = x - y$

7. $(p^2 + q^2)y = qz$

Solution The given equation is

① $$q - p + x - y = 0$$

$$p - x = -y + q = a, \text{ say}$$

$\therefore$ $p - x = a$ and $q - y = a$

$\therefore$ $p = x + a$ and $q = y + a$

Putting these values of p and q in

$dz = p\,dx + q\,dy$

$= (x+a)\,dx + (y+a)\,dy$

Integrating, we get

$$z = \frac{(x+a)^2}{2} + \frac{(y+a)^2}{2} + \text{constant}$$

$$2z = (x+a)^2 + (y+a)^2 + \text{constant} (= b)$$

$$= (x+a)^2 + (y+a)^2 + b.$$

Solution The given equation is

② $$\sqrt{p} + \sqrt{q} = 2x$$

$$\sqrt{p} - 2x = -\sqrt{q} = a, \text{ say}$$

$\therefore$ $\sqrt{p} - 2x = a$ and $-\sqrt{q} = a$

$\therefore$ $\sqrt{p} = 2x + a$ and $\sqrt{q} = -a$

$\therefore$ $p = (2x+a)^2$ and $q = a^2$

Putting these values of p and q in

$dz = p\,dx + q\,dy$

$= (2x+a)^2\,dx + a^2\,dy$

Integrating, we get

$$z = \frac{(2x+a)^3}{2 \times 3} + a^2 y + b$$

$$= \frac{(2x+a)^3}{6} + a^2 y + b$$

Solution: The given equation is

③ $$q = xp + p^2$$

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tion: The given equation is

(4) $$z^2(p^2+q^2) = x^2+y^2$$

$$z^2\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right] = x^2+y^2$$

Here in the question it is to be noted that z is directly present in the given differential equation. Hence, we need certain substitution to reduce it in the standard form IV.

Put $z\partial z = \partial Z_1$

$$\left(z\frac{\partial z}{\partial x}\right)^2+\left(z\frac{\partial z}{\partial y}\right)^2 = x^2+y^2$$

$$\therefore \left(\frac{\partial Z_1}{\partial x}\right)^2 - x^2 = y^2-\left(\frac{\partial Z_1}{\partial y}\right)^2$$

$$\therefore P^2-x^2 = y^2-Q^2, \quad P = \frac{\partial Z_1}{\partial x} \text{ and } Q = \frac{\partial Z_1}{\partial y}$$

Which is a standard form IV.

$$\therefore P^2-x^2 = y^2-Q^2 = a^2, \text{ say}$$

$$\therefore P^2 = x^2+a^2 \text{ and } Q^2 = y^2-a^2$$

$$\therefore P = \sqrt{a^2+x^2} \text{ and } Q = \sqrt{y^2-a^2}$$

Putting these values of P and Q in

$$dZ_1 = P\,dx + Q\,dy$$

$$z\partial z = \sqrt{a^2+x^2}\,dx + \sqrt{y^2-a^2}\,dy$$

Integrating, we get

$$\frac{z^2}{2} = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right) +$$

$$\frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\log\left(y+\sqrt{y^2-a^2}\right) + b$$

$$\because \int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$$

$$\int\sqrt{y^2-a^2}\,dy = \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\log\left(y+\sqrt{y^2-a^2}\right)$$

$$z^2 = x\sqrt{x^2+a^2} + a^2\log\left(x+\sqrt{x^2+a^2}\right) + y\sqrt{y^2-a^2}$$

$$- a^2\log\left(y+\sqrt{y^2-a^2}\right) + 2b$$

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Solution: The given differential equation is
(5)
$$z(p^2 - q^2) = x - y$$
$$\left(z^{1/2}\frac{\partial z}{\partial x}\right)^2 - \left(z^{1/2}\frac{\partial z}{\partial y}\right)^2 = x - y$$
$$\left(z^{1/2}\frac{\partial z}{\partial x}\right)^2 - x = \left(z^{1/2}\frac{\partial z}{\partial y}\right)^2 - y$$
Put $z^{1/2}\partial z = \partial z_1$ or $z^{1/2}dz = dz_1$
$$\therefore \left(\frac{\partial z_1}{\partial x}\right)^2 - x = \left(\frac{\partial z_1}{\partial y}\right)^2 - y$$
$$\therefore P^2 - x = Q^2 - y \quad , P = \frac{\partial z_1}{\partial x} , Q = \frac{\partial z_1}{\partial y}$$
Let $P^2 - x = Q^2 - y = a$, say
$\therefore P^2 = x + a$ and $Q^2 = y + a$
$\therefore P = \sqrt{x+a}$ and $Q = \sqrt{y+a}$
Putting these values of P and Q in
$dz_1 = P\,dx + Q\,dy$
$z^{1/2}dz = (x+a)^{1/2}dx + (y+a)^{1/2}dy$
Integrating, we get
$$\frac{z^{3/2}}{3/2} = \frac{(x+a)^{3/2}}{3/2} + \frac{(y+a)^{3/2}}{3/2} + \text{constant}$$
$$z^{3/2} = (x+a)^{3/2} + (y+a)^{3/2} + \text{constant } (= b)$$
$$z^{3/2} = (x+a)^{3/2} + (y+a)^{3/2} + b.$$

Solution The given differential equation is
(6)
$$p^2 - q^2 = x - y$$
$$p^2 - x = q^2 - y = c, \text{ say}$$
$$p^2 - x = c \text{ and } q^2 - y = c$$
$$p^2 = x + c \text{ and } q^2 = y + c$$
$$p = (x+c)^{1/2} \text{ and } q = (y+c)^{1/2}$$
Putting these values of p and q in
$dz = p\,dx + q\,dy$

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8. Tick ✓ the correct answer.

(a) The Partial differential equation from
$$z = (a+x)^2 + y$$
is .....

(i) $z = \frac{1}{4}\left(\frac{\partial z}{\partial x}\right)^2 + y$ (ii) $z = \frac{1}{4}\left(\frac{\partial z}{\partial y}\right)^2 + y$

(iii) $z = \left(\frac{\partial z}{\partial x}\right)^2 + y$ (iv) $z = \left(\frac{\partial z}{\partial y}\right)^2 + y$

Ans:- $z = \frac{1}{4}\left(\frac{\partial z}{\partial x}\right)^2 + y$

(b) The solution of $xp + yq = z$ is ....

(i) $f(x+y, y+\log z) = 0$ (ii) $f(xy, y\log z) = 0$

(iii) $f(x-y, y-\log z) = 0$ (iv) None of these.

Ans:- None of these.

(c) The solution of $xp + yq = z$ is ....

(i) $f(x, y) = 0$ (ii) $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$

(iii) $f(xy, yz) = 0$ (iv) $f(x^2, y^2) = 0$

Ans:- $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

(d) The solution of $p + q = z$ is ....

(i) $f(x+y, y+\log z) = 0$ (ii) $f(xy, y\log z) = 0$

(iii) $f(x-y, y-\log z) = 0$ (iv) None of these

Ans:- $f(x-y, y-\log z) = 0$

(e) The solution of
$$(y-z)p + (z-x)q = x-y$$
is ...

(i) $f(x+y+z) = xyz$ (ii) $f(x^2+y^2+z^2) = xyz$

(iii) $f(x^2+y^2+z^2, x^2y^2z^2) = 0$ (iv) $f(x+y+z) = x^2+y^2+$

Ans:- $f(x+y+z) = x^2+y^2+z^2$.

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$dz = (x+c)^{1/2} dx + (y+c)^{1/2} dy$

Integrating, we get

$z = \frac{2}{3}(x+c)^{3/2} + \frac{2}{3}(y+c)^{3/2} + c_1$

Solution: ⑦ The given differential equation is

$(p^2+q^2)y = qz$

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