Q1 Attempt any five of the following.

(a) Define dictionary order relation on $A \times B$.

⇒ Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation on $A \times B$ by defining

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$. or if $a_1 = a_2$ and $b_1 <_B b_2$. It is called the dictionary order relation on $A \times B$.

(b) Define Composite mapping.

⇒ Composite Mapping :-
Given function $f: A \to B$ and $g: B \to C$, we define the Composite g of of f and g as the function.

$g \circ f : A \to C$ define the equation

$$g \circ f(a) = g(f(a))$$

formally, $g \circ f : A \to C$ is the function whose rule is

$$\{(a, c) \mid \text{for some } b \in B,\ f(a) = b \ \& \ g(b) = c\}$$

(c) Define open ray & close ray.

⇒ open Ray :-

ⓓ Define Homeomorphism.

⇒ Homeomorphism :-

Let x and y be topological spaces.

Let $f: x \longrightarrow y$ be a bijection. If both the function f and the inverse function

$f^{-1}: y \rightarrow x$. are continuous then f is called homeomorphism

ⓔ Define Connected space.

⇒ Connected space:-

Let x be a topological space. A seperation of x is a pair U, V of disjoint non-empty open subset of x whose union is x. The space x is said to be conneted if there does not exist a seperation of x.

ⓕ Define open cover.

⇒

Q.2) Attempt any two of the following.

@ If there exist an uncountable well-ordended set, then prove that every section of which is Countable.

⇒ Proof :-
Let we begin with an uncountable well ordered set B.

Let C be the well-ordered set $\{1,2\} \times B$ in the dictionary order. Then
Some section of C is uncountable.
Let,
Ω be the smallest element of C for which the section of C by Ω is uncountable then let A consist of this section along with the element Ω.

S_Ω is an uncountable well-ordered set every section of which is countable.

It is order type is in fact uniquely determined by this condition.
Call it a minimal uncountable well ordered set

set $A = S_\Omega \cup \{\Omega\}$

Hence is countable

Hence proved.

(5) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ & $\mathcal{T}'$ respectively, on X Then prove that the following statements are equivalent.

(i) $\mathcal{T}'$ is linear than $\mathcal{T}$.

(ii) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x. There is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

⇒ Proof :-

(2) = (1)

Given an element U of $\mathcal{T}$.

we wish to show that $U \in \mathcal{T}'$

let $x \in U$, Since $\mathcal{B}$ generates $\mathcal{T}$, There is an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

Condition (1) tells us there exist an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$ Then,

$x \in B' \subset U$, so $U \in \mathcal{T}$ by definition.

(1) = (2) , we are given $x \in X$ and $B \in \mathcal{B}$ with $x \in B$

Now $B \in \mathcal{T}$ by definition & $\mathcal{T} \subset \mathcal{T}'$ by Condition (1)

Therefore $B \in \mathcal{T}'$

Since $\mathcal{T}'$ is generated by $\mathcal{B}'$

There is an element $B' \in \mathcal{B}'$

such that $x \in B' \subset B$

Hence proved.

Q.3 Attempt any two of the following.

(a) If B is a basis for the topology of x and C is a basis for the topology of Y then prove that the Collection

$D = \{ B \times C / B \in \mathcal{B} \text{ and } C \in \mathcal{C} \}$ is a basis for the topology of $X \times Y$.

⇒ Proof :-

By using previous lemma, Given an open set W of $X \times Y$ and a point $x \times y$ of W.

By defination of the product topology there is a basis element $U \times V$ such that.

$x \times y \in U \times V \subset W$

Because B and C are bases for x and y respectively. We can choose an element B of $\mathcal{B}$ such that $x \in B \subset U$, and an element C of $\mathcal{C}$ such that

$y \in C \subset V$

Then

$x \times y \in B \times C \subset W$

Thus the Collection D meets the Criterion of lemma

∴ So D is a basis for $X \times Y$.

Hence proved.

(b) Define Subspace topology & Prove that let Y be a Subspace of X. If U is open in Y and Y is open in X. Then U is open in X.

⇒ Subspace Topology :- Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X the collection

$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$ is a topology on Y called the subspace topology.

Proof :-

Since U is open in Y.
$U = Y \cap V$ for some set V open in X.

Since Y and V are both open in X

So is $Y \cap V$

i.e. U is open in X

Hence proved.

Q.4

(a) Let X be a topological space, then prove that the following conditions holds

(i) ϕ & X are closed.

(ii) Arbitrary intersections of closed sets are closed.

(iii) Finite unions of closed sets are closed.

⇒ Proof :-

Let X be a topological space.

then

(i) ϕ and X are closed because they are complements of the open sets X & ϕ respectively.

(ii) Given a collection of closed sets $\{A_\alpha\}_{\alpha \in J}$ we apply De-morgans law

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$$

Since the sets $X - A_\alpha$ are open by defination the right sides of this equation represents an arbitrary union of open sets and is thus open therefore

$\bigcap A_\alpha$ is closed.

(iii) Similarly, If A_i is closed for $i = 1, \ldots, n$.

Consider the equation

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i).$$

The set on the right side of this equation is a finite intersection of open sets & is therefore open

Hence $\bigcup A_i$ is closed.

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(b) Prove that, let A be a subset of the topological space; let A' be the set of all limit points of A then $\bar{A} = A \cup A'$.

⇒ Proof :-

If x in A', every nbd of x intersects A.
Therefore, by previous theorem.
$x \in \bar{A}$.

Hence $A' \subset \bar{A}$.

Since by defination $A \subset \bar{A}$.
it follows that $A \cup A' \subset \bar{A}$.

To demonstrate the reverse inclusion.
We let, x be a point of $\bar{A}$ and show that $x \in A \cup A'$.

If x happens to lie in A,
it is trivial that $x \in A \cup A'$ --- (1)
Suppose that x does not lie in A,
Since,
$x \in \bar{A}$, we know that every nb'd U of x intersets A

because $x \notin A$. The set U must intersects A in a point different from x
then $x \in A'$ ---- (2)

So that $x \in A \cup A'$ is decides by equation (1) & (2).

$$\boxed{\therefore \bar{A} = A \cup \bar{A}}$$

∴ Hence proved.

(c) Prove that the product of two Hausdorff spaces is Hausdorff.

⇒ Proof :-

Q.1 Attempt any five of the following.

(a) Define well ordered set.

⇒ Well ordered set :-

A set A with an Order relation $<$ is said to be well-ordered if every non-empty subset of A has a smallest element e.g. The set $\{1, 2\} \times 2$ is the well ordered set.

(b) Define. Discreate Topology.

⇒ Discrete Topology :-

(c) Define interior of a set.

⇒

Given a subset A of a topological space X, the interior of A is defined as the union of all open sets contained in A and the closure of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by Int A and the closure of A is denoted by cl A or by $\bar{A}$.

(d) Define Hausdorff spaces.

⇒ Hausdorff spaces :- A topological space X is called a Hausdorff space If for each pair x_1, x_2 of a distinct points of X.

There exist neighborhoods U_1 & U_2 of x_1 & x_2 respect-ively that are disjoint.

(e) Define a Connected Space.

→ Connected Space :-
Let x be a topological space A seperation of x is a pair u, v of disjoint non-empty open subset of x whose union is x. The space x is said to be connected, If there does not exist a seperation of x.

Also, A space x is connected if and only if the only subsets of x that are ~~empty~~ both open and closed in x are the empty set and x itself.

(f) Define a basis for a Topology.

⇒ If x is a set, a basis for a topology on x is a collection B of subsets of x called basis elements such that

(i) for each $x \in X$, there is at least a basis element B containing x.

(ii) If x belongs to the intersection two basis elements B_1 & B_2 then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$

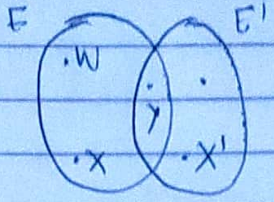
If B satisfies these two conditions then we define the topology T generated by B as follows

A subset U of x is said to be open in X (i.e. to be an element of T) If for each $x \in U$, there is a basis element $B \in B$ such that $x \in B$ & $B \subset U$.

Q.2 Attempt any two of the following.

(a) Show that two equivalence classes E & E' are either disjoint or equal.

⇒ Proof :-



Let E be the equivalence class determined by x and let E' be the equivalence class determined by x'.

Suppose that $E \cap E'$ is not empty let y be a point $E \cap E'$ (as shown above)

To Prove that $E = E'$

By definition we have,

$y \sim x$ & $y \sim x'$

Symmetry allows us to conclude that $x \sim y$ & $y \sim x'$

∴ from transistivity it follows that $x \sim x'$.

If now w is any point of E, we have $w \sim x$ by defination. It follows from another application of transistivity that $w \sim x'$.

We conclude that $E \subset E'$.

The symmetry of the situation to conclude that

$E' \subset E$

∴ $E = E'$

Given an equivalence to relation on set A let us denot by $\mathcal{E}$ the collection of all the equivalence classes determined by this relation. It is shows that distinct element of $\mathcal{E}$ are disjoint.

(b) Define any five topologies on $X = \{a, b, c\}$.

$\Rightarrow$

ⓒ If {Ta} is family of topologies on x, show that ∩Ta is a topology on x is ∪Ta topology on x?
It then Justify.

Q.3 Attempt any two of the following.

(a) Prove that the Collection

$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$ is

a Sub basis for the product topology on $X \times Y$.

⇒ Proof :-

Let $\mathcal{T}$ denote the product topology on $X \times Y$.

Let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$.

Because every element of $\mathcal{S} \in \mathcal{T}$.

So do arbitrary unions of finite intersections of elements of $\mathcal{S}$

Thus

$\mathcal{T}' \subset \mathcal{T}$

On the other hand, every element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of $\mathcal{S}$.

Since,

$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$

$\therefore$ $U \times V \in \mathcal{T}'$

So that $\mathcal{T} \subset \mathcal{T}'$ as well.

(b) If A is a subspace of X & B is a subspace of Y, then prove that the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

⇒ Proof :-

The set $U \times V$ is general basis element for $X \times Y$, where U is open in X and V is open in Y.

Therefore

$(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$.

Now,

$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$

Since,

$U \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A & B respectively.

The set $(U \cap A) \times (V \cap B)$ is the general basis element for the product Topology on $A \times B$.

So that the subspace topology on $A \times B$ & for the product topology $A \times B$ are the same.

Hence proved.

(c) Consider the following subset of the real line $Y = [0,1] \cup (2,3)$ prove that both $[0,1]$ & $(2,3)$ are closed as subset of Y.

⇒ Proof :-

If this space, the set $[0,1]$ is open, since it is the intersection of the open set $\left(-\frac{1}{2}, \frac{3}{2}\right)$ of R with Y.

Similarly,
$(2,3)$ is open as a subset of Y It is even open as a subset of R

Since,
$[0,1]$ & $(2,3)$ are Complements in Y of each other.

we Conclude that both $[0,1]$ and $(2,3)$ are Closed as subset of Y

Hence proved.

(C) Consider the following subset of the real line $Y = [0,1] \cup (2,3)$ prove that both $[0,1]$ and $(2,3)$ are closed as subset of Y.

⇒ Proof :-

In this space, the set $[0,1]$ is open, since it is the intersection of the open set $(-\frac{1}{2}, \frac{3}{2})$ of R

with Y.

Q.3 Attempt any two of the following.

(a) Let Y be a subspace of X. Then proved that a set A is closed in Y iff it equal the intersection of a closed set of X with Y.

⇒ Proof :-

Assume that,

$A = C \cap Y$, where C is closed in X.

then,

$X - C$ is open in X, so that $(X-C) \cap Y$ is open in Y. is open in Y

By the definition of subspace topology

But $(X-C) \cap Y = Y - A$

Hence, $Y - A$ is open in Y.

So that A is closed in Y.

Conversly :-

Assume that A is closed. in Y Then $Y - A$ is open in Y so that by definition : It equals the intersection of an open set U of X with Y.

The set $X - U$ is closed in X and $A = Y \cap (X - U)$

So that A equals the intersection of a closed set of X with Y

Hence proved.

(c) Let $f: A \rightarrow X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$ then prove that f is continuous iff the functions:
$f_1: A \rightarrow X$ and $f_2: A \rightarrow Y$ are continuous

⇒ Proof :- Let
$\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ be projection onto the first and second factors respectively.
These maps are continuous
for $\pi_1^{-1}(U) = U \times Y$ and
$\pi_2^{-1}(V) = X \times V$ and these sets are open if U and V are open for each $a \in A$.

$\therefore$ $f_1(a) = \pi_1(f(a))$ and
$f_2(a) = \pi_2(f(a))$.
Also, If the function f is continuous then f_1 & f_2 are compositon of continuous function & therefore continuous.
Conversly :- Suppose that f_1 & f_2 are continuous we show that for each basis element $U \times V$ for the topology of $X \times Y$.

$\therefore$ It is inverse image $f^{-1}(U \times V)$ is open
A point a is in $f^{-1}(U \times V)$ It is
$f(a) \in U \times V$ i.e. iff $f_1(a) \in U$ & $f_2(a) \in V$
$\therefore$ $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$

Since,
both the sets $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open, so is their intersection.

$\therefore$ If $: A \times B \rightarrow X$ whose domain is a product space.
$\therefore$ f is continuous if it is continuous in each variable seperadely.
But this is contradiction.
Hence proved.