

Numerical Analysis

Q.1 Attempt any five of the following.

(a) Define Argument and Entry.

⇒ Argument :-

If $y = f(x)$ be a function then the values of independent variable x are called Arguments.

denoted by $x_0, x_1, x_2, \dots, x_n$.

Entry :-

The values of dependent variable y corresponding to the argument is called entry.

denoted by $y_0, y_1, y_2, \dots, y_n$.

(b) Evaluate : $\Delta^2 x^3$.

⇒

① Define numerical quadrature :-

Numerical quadrature :-

We observe integrals which cannot be easily or exactly evaluated. If the values of integral for few values of variables are given, we cannot integrate. Then the numerical method or integration are used.

Definition :-

The process of finding and evaluating a definite integral from a set of numerical values of integrand, when applied to integration of function is known as quadrature.

② State trapezoidal rule as approximate quadrature formula
 $\Rightarrow$ Trapezoidal Rule :-

$$\int_{x_0}^{x_0+nh} y dx = \frac{1}{4} \left[\frac{y_0 + y_n}{2} + y_1 + y_2 + y_3 + \dots + y_{n-1} \right]$$

Called as trapezoidal Rule.

Q.2 Attempt any two of the following.

(a) Prove that n^{th} divided differences of a polynomial of the n^{th} degree are constant.

$\Rightarrow$ Proof :-

The rational integral function (Polynomial) of n^{th} degree in x be

$$\therefore f(x) = ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l$$

where, n is a positive integer $a, b, c, \dots, k, l$ are constant and $a \neq 0$

$$f(x+h) = a(x+h)^n + b(x+h)^{n-1} + c(x+h)^{n-2} + \dots + kx + l$$

Now,

$$\begin{aligned} \Delta f(x) &= f(x+h) - f(x) \\ &= [a(x+h)^n + b(x+h)^{n-1} + \dots + kx + l] \\ &\quad - [ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l] \end{aligned}$$

$$\therefore \Delta f(x) = a_n b x^{n-1} + b' x^{n-2} + c' x^{n-3} + \dots + k' x + l' \dots \text{--- (1)}$$

Where,

b', c', k', l' are constant thus the 1^{st} difference of rational integral function (Polynomial) is polynomial of degree $n-1$.

So,

Second divided difference of $f(x)$ is

$$\begin{aligned} \Delta^2 f(x) &= \Delta(\Delta f(x)) \\ &= \Delta(f(x+h) - f(x)) \\ &= \Delta f(x+h) - \Delta f(x) \end{aligned}$$

$$= a_n h(x+h)^{n-1} + b'(x+h)^{n-2} + c'(x+h)^{n-3} + \dots + k(x+h) + l$$

$$- (a_n h x^{n-1} + b' x^{n-2} + c' x^{n-3} + \dots + k' x + l')$$

$$= a_n(n-1) h^2 x^{n-2} + b' x^{n-3} + c' x^{n-4} + \dots + k' x + l'$$

The second difference of the n^{th} degree polynomial is a polynomial of degree $n-2$.

By Continuing in this manner we arrive at the result

$$\Delta^n f(x) = a[n(n-1)(n-2) \dots 2 \cdot 1] h^n$$

$$= a h^n n!$$

$$\Delta^n f(x) = a[n(n-1)(n-2) \dots 2 \cdot 1] h^n$$

$$= a h^n n!$$

So the n^{th} difference is a constant

Hence proved.

(b) Evaluate $\Delta^n (e^{ax+b})$. where $a \neq b$ are constants and interval of differentiating is unity.

$\Rightarrow$

We know that,

$$\Delta f(x) = f(x+h) - f(x)$$

$$\therefore \Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$$

$$= e^{ax+ab} - e^{ax+b}$$

$$= e^{ax+b+a} - e^{ax+b}$$

$$= e^{ax+b} (e^a - 1)$$

Similarly,

$$\Delta^2 e^{ax+b} = \Delta [\Delta e^{ax+b}]$$

$$= \Delta [e^{ax+b} (e^a - 1)]$$

$$= e^{a(x+h)+b} (e^a - 1) - e^{ax+b} (e^a - 1)$$

$$= e^{ax+ab} (e^a - 1) - e^{ax+b} (e^a - 1)$$

$$= e^{ax+b+a} (e^a - 1) - e^{ax+b} (e^a - 1)$$

$$= (e^{ax+b+a} - e^{ax+b}) (e^a - 1)$$

$$= [e^{ax+b} (e^a - 1)] (e^a - 1)$$

$$= e^{ax+b} (e^a - 1)^2$$

~~Similarly~~

Similarly,

$$\Delta^3 e^{ax+b} = e^{ax+b} (e^a - 1)^3$$

Continuing this process we get

$$\Delta^n e^{ax+b} = e^{ax+b} (e^a - 1)^n$$

(C) From the following table, Find the number of students who obtained less than 45 marks.

Marks	Number of Students
30-40	31
40-50	42
50-60	51
60-70	35
70-80	31

⇒ Solution:-

Max mark less than (a)	No. of students	Δ	Δ^2	Δ^3	Δ^4
40	31	42	9	25	37
50	73	31	16	12	
60	124	35	4		
70	159	31			
80	190				

Here, $h=10$, $x=45$, $a=40$

$$u = \frac{x-a}{h} = \frac{45-40}{10} = \frac{5}{10} = 0.5$$

$\therefore$ By newton Forward interpolation formula.

$$P_n(x) = f(a) + u \Delta f(a) + \frac{u(u-1)\Delta^2 f(a)}{2!} + \frac{u(u-1)(u-2)\Delta^3 f(a)}{3!} + \frac{u(u-1)(u-2)(u-3)\Delta^4 f(a)}{4!} + \dots$$

$$\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)\Delta^4 f(a)}{4!} + \dots$$

$$P_7(x) = 40 + (0.5)31 + (0.5)(0.5-1)(-4) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)(-12)}{3 \times 2} + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)(-37)}{4 \times 3 \times 2} + \dots$$

$$\frac{(0.5)(0.5-1)(0.5-2)(0.5-3)(-12)}{3 \times 2} + (0.5)$$

$$\frac{(0.5)(0.5-1)(0.5-2)(0.5-3)(-37)}{4 \times 3 \times 2}$$

$$P_7(x) = 40 + 15.5 + 0.5 + 0.75 - 1.44$$

$$P_7(x) = 56.75 - 1.44$$

$$\therefore P_7(x) = 55.3$$

Q.3

a) Prove that Newton's formula for unequal intervals.

⇒ Proof :-

Let $f(x_0), f(x_1), f(x_2) \dots f(x_n)$ be the values of $f(x)$ corresponding to the arguments.

$x_0, x_1, x_2 \dots x_n$ not equally spaced

∴ By the definition of divided difference

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad \dots \textcircled{I}$$

Now

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1} \quad \dots \textcircled{II}$$

$$\therefore f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1) \quad \dots \textcircled{III}$$

Similarly,

$$f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2) \quad \dots \textcircled{IV}$$

⋮

$$f(x, x_0, x_1, x_2) = f(x_0, x_1, x_2, x_3) + (x - x_3) f(x, x_0, x_1, x_2, x_3) \quad \dots \textcircled{V}$$

Similarly,

$$f(x, x_0, x_1, x_2, \dots, x_{n-1}) = f(x_0, x_1, x_2, \dots, x_n) + (x - x_n) f(x, x_0, x_1, \dots, x_n) \quad \textcircled{VI}$$

Multiply I' by $x - x_0$

I'' by $(x - x_0)(x - x_1)$ and

I''' by $(x - x_0)(x - x_1)(x - x_2)$ and

Iⁿ by $(x - x_0)(x - x_1) \dots (x - x_n)$ and

add it (I) we get,

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \dots + (x - x_0)(x - x_1) \dots (x - x_n)f(x_0, x_1, \dots, x_n) +$$

where

$$R_n = (x - x_0)(x - x_1) \dots (x - x_n)f(x_0, x_1, \dots, x_n)$$

Assuming that $f(x)$ is polynomial of n

So,

divided difference $f(x_0, x_1, \dots, x_n)$ vanish is zero

Hence,

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \dots + (x - x_0)(x - x_1) \dots (x - x_n)f(x_0, x_1, \dots, x_n)$$

This is called Newton's divided difference interpolation formula for unequal interval.

(b) Prove Bessel's interpolation formula.

⇒ Proof :-

we know that, the Gauss forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_{-1} + \frac{u(u^2-1)}{3!} \Delta^3 y_{-1}$$

$$+ \frac{u(u^2-1)(u^3-2)}{4!} \Delta^4 y_{-2} + \dots$$

The gauss backward formula with origin transfer to unity takes the form.

$$y_u = y_1 + {}^u C_1 \Delta y_0 + {}^u C_2 \Delta^2 y_0 + {}^u C_3 \Delta^3 y_0 + \dots$$

Taking the mean of this, with gauss's forward formula is

$$y_u = \frac{1}{2} (y_0 + y_1) + (u - \frac{1}{2}) \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \dots$$

(C) Interpolate by means of gauss backward formula the population for the year 1936. Given the following table

years	Population (1000)
1901	12
1911	15
1921	20
1931	27
1941	39
1951	52

⇒ Solution :-

$$\text{let } u = \frac{x - x_0}{h}$$

$$\text{here } , x = 1936 , h = 10$$

Take the origin at 1931

$$u = \frac{1936 - 1931}{10} = 0.5$$

The difference table is

u	y _u	Δy _u	Δ ² y _u	Δ ³ y _u	Δ ⁴ y _u	Δ ⁵ y _u
-3	12	3	2	0	3	-10
-2	15	5	2	3	7	
-1	20	7	5	4		
0	27	12	1			
1	39	13				
2	52					

The gauss

The Gauss backward formula is

$$\begin{aligned}
 y_u &= y_0 + u \Delta y_{-1} + \frac{u(u+1)}{2!} \Delta^2 y_{-1} + \frac{u(u+1)(u+2)}{3!} \Delta^3 y_{-2} \\
 &\quad + \frac{u(u+1)(u+2)(u+3)}{4!} \Delta^4 y_{-3} + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!} \Delta^5 y_{-4} \\
 &= 27 + 0.5 \times 7 + \frac{0.5 \times 1.5 \times 2}{1 \times 2} + \frac{0.5 \times 0.75 \times 3}{1 \times 2 \times 3} + \frac{(0.5)(0.75)(2.5)}{1 \times 2 \times 3 \times 4} \\
 &\quad + \frac{(0.5)(0.75)(-3.75)(-10)}{1 \times 2 \times 3 \times 4 \times 5} \\
 &= 27 + 3.5 + 0.75 - 0.1875 + 0.11718 - 0.11718 \\
 &= 31.25 - 0.1875 \\
 &= 31.0625
 \end{aligned}$$

Q4

(a) Prove the general quadrature formula for equidistant ordinates.

⇒ proof :-

$$I = \int_0^b y \cdot dx$$

where, $y = f(x)$

Suppose,

$f(x)$ is a given for certain equidistant values of x
 say $x_0, x_0+h, x_0+2h, \dots, x_0+nh$
 let,

The range (a,b) be divided into n equal parts, each of width h so that,

$$b-a = nh$$

$$\text{let, } x_0 = a, \quad x_1 = a+h, \quad x_2 = a+2h \\ x_n = a+nh = b$$

So, we assume that $n+1$ ordinates $y_0, y_1, y_2, \dots, y_n$ are equidistant

$$\text{So, } I = \int_a^b y \cdot dx$$

$$= \int_{x_0}^{x_0+nh} y \cdot dx$$

$$= \int_0^n y_{x_0+hu} \cdot h \cdot du$$

Where,

$$u = \frac{x - x_0}{h} \quad \therefore dx = h \cdot du$$

Using Newton's backward formula,

$$y = y_{x_0} + n h u$$

$$P_n(x) = P_n(x_0 + nu)$$

$$I = \int_0^n [y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \dots] h \cdot du$$

$$= h \left[y_0 \left[\frac{u^2}{2} \right]_0^n + \frac{\Delta y_0}{2!} \left[\frac{u^3}{3} - \frac{u^2}{2} \right]_0^n + \frac{\Delta^2 y_0}{3!} \left[\frac{u^4}{4} - \frac{3u^3}{3} + \frac{3u^2}{2} - u \right]_0^n + \dots \right]$$

$$I = h \left[n y_0 + \frac{n^2}{2} \Delta y_0 + \frac{n^3}{3} - \frac{n^2}{2} \Delta^2 y_0 + \frac{n^4 - n^3 + n^2}{3!} \Delta^3 y_0 + \dots \right]$$

+ upto nth term

This is general quadrature formula.

- (b) Apply Simpson's rule to estimate the value of the integral : $\int_1^2 \frac{dx}{x}$

By dividing the intervals (1,2) into 4 equal parts.

⇒ Solutions :

Here,

$$\therefore y = f(x) = \frac{1}{x}$$

$$a=1, \quad b=2, \quad h=\frac{1}{4}$$

$$h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

$$x: 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$$

$$y: 1, \frac{4}{5}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}$$

By Simpson's $\frac{1}{3}$ rd rule is

$$\int_a^b y dx = \frac{1}{3} \times h [(y_0 + y_n) + 2(y_2 + y_4 + \dots + y_{n-2}) + 4(y_1 + y_3 + \dots + y_{n-1})]$$

$$\int_1^2 \frac{dx}{x} = \frac{1}{4/3} \left[\left(1 + \frac{1}{2}\right) + 2\left(\frac{2}{3}\right) + 4\left(\frac{4}{5} + \frac{4}{7}\right) \right]$$

$$= \frac{1}{12} \left[\frac{3}{2} + \frac{4}{3} + 5.4857142856 \right]$$

$$= \frac{1}{12} [2.833333 + 5.485142850]$$

$$= \frac{1}{12} \times 8.3190476189$$

$$= 0.69325395$$

$$= 0.77 \text{ Approximately}$$

By Simpson's $\frac{3^{th}}{8}$ rule

$$\int_a^b y dx = \frac{3}{8} \times h [y_0 + y_n] + 3[y_1 + y_2 + y_n + y_5 + y_7 + y_8 \dots +$$

$$y_{n-2} + y_{n-1}] + 2[y_3 + y_6 + y_9 + \dots + y_{n-3}]$$

$$\int_1^2 \frac{dx}{x} = \frac{3}{8} + \frac{1}{9} [C(y_0 + y_n) + 3(Cy_1 + y_n) + 2(Cy_1)]$$

$$= 0.09375 \left[\left(1 + \frac{1}{2}\right) + 3\left(\frac{4}{5} + \frac{2}{3}\right) + 2\left(\frac{4}{5}\right) \right]$$

$$= 0.00$$

$$= 0.09375 \times 7.0428571429$$

$$= 0.660267$$

© Given:

$$\frac{dy}{dx} = \frac{y-x}{y+x} \quad \text{with the boundary condition } y=1 \text{ for } x=0$$

Find approximately for $x=0.1$ by Euler's method (two steps)

⇒ Solution:-

We have initial condition

$$y=1 \text{ for } x=0$$

$$x_0=0, y_0=1 \text{ and } n=5 \text{ } x=0.1$$

$$\Delta x = h = \frac{x - x_0}{n} = \frac{0.1}{5} = 0.02$$

Step I:- By Euler's method

$$y_1 = y_0 + f(x_0, y_0)$$

Here,

$$\frac{dy}{dx} = \frac{y-x}{y+x} = f(x, y)$$

$$f(x_0, y_0) = \frac{y_0 - x_0}{y_0 + x_0} = \frac{1-0}{1+1} = 1$$

$$y_1 = 1 + (0.02 \times 1)$$

$$y_1 = 1.02$$

Step II :- $y_2 = y_1 + hf(x_1, y_1)$

Here

$$y_1 = 1.02, \quad x_1 = x_0 + h = 0 + 0.02 = 0.02$$

$$f(x_1, y_1) = \frac{y_1 - x_1}{y_1 + x_1} = \frac{1.02 - 0.02}{1.02 + 0.02} = \frac{1}{1.04}$$

$$= 0.96153$$

$$\therefore y_2 = y_1 + hf(x_1, y_1)$$

Step III :-

By Euler's method

$$y_3 = y_2 + hf(x_2, y_2)$$

Here,

$$y_2 = 1.0392, \quad x_2 = x_1 + h = 0.02 + 0.02 = 0.04$$

$$f(x_2, y_2) = \frac{y_2 - x_2}{y_2 + x_2} = \frac{1.0392 - 0.04}{1.0392 + 0.04}$$

$$= \frac{0.9992}{1.0792}$$

$$= 0.92584$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 1.0392 + (0.02)(0.92584)$$

$$y_3 = 1.0577$$

Step IV :- By Euler's method.

$$\text{Here, } y_4 = y_3 + hf(x_3, y_3)$$

$$y_3 = 1.0577, \quad x_3 = x_2 + h = 0.04 + 0.02 = 0.06$$

$$f(x_3, y_3) = y_3 - x_3 = 1.0577 - 0.06$$

$$f(x_3, y_3) = y_3 + x_3 = 1.0577 + 0.06$$

$$\frac{f(x_3, y_3)}{f(x_3, y_3) + f(x_3, y_3)} = \frac{0.9977}{1.1177}$$

$$x_4 = 0.8926$$

(10, 10) 7.111 1.111

2.111 1.111

(bottom value) 1.111

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Q.1

(a) Prove that $E\Delta = \Delta E$ $\Rightarrow$ Proof :-

$$\begin{aligned}
 E\Delta f(x) &= E(f(x+h) - f(x)) \\
 &= Ef(x+h) - Ef(x) \\
 &= f(x+2h) - f(x+h) \\
 &= \Delta f(x+h) \\
 &= \Delta Ef(x)
 \end{aligned}$$

$$\therefore E\Delta = \Delta E$$

(b) Define third divided difference of $f(x)$ for arguments x_0, x_1, x_2, x_3 $\Rightarrow$

$$\begin{aligned}
 \Delta_{x_1, x_2, x_3}^3 f(x_0) &= \Delta_{x_1, x_2, x_3}^1 f(x_0) = f(x_0, x_1, x_2, x_3) \\
 &= f(x_0, x_1, x_2) - (f(x_1, x_2, x_3)) \\
 &\quad x_0 - x_3
 \end{aligned}$$

(c) Prove that $\Delta^3 = \Delta E^2$ $\Rightarrow$ Proof :-

$$f = \frac{1}{12} \left(\frac{1}{12} + \frac{1}{12} + \frac{1}{12} \right)$$

$$\frac{1}{12} \left(\frac{1}{12} + \frac{1}{12} + \frac{1}{12} \right)$$

(d) Define difference product of x_1, x_2, x_3 in terms of determinants. $\Rightarrow$

(e) Define Numerical Quadrature

⇒

(f) State Weddles rule as approximate quadrature formula.

⇒ The general quadrature formula is

$$\int_{x_0}^{x_0+nh} y dx = h \left[ny_0 + \frac{n^2}{2} \Delta y_0 + \left(\frac{n^3}{8} - \frac{n^2}{2} \right) \Delta^2 y_0 + \right.$$

$$\left(\frac{n^4}{4} + n^3 + n^2 \right) \frac{\Delta^3 y_0}{3!} +$$

$$\left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11}{3} n^3 - 3n^2 \right) \frac{\Delta^4 y_0}{4!}$$

Then

$$\int_{x_0}^{x_0+nh} y dx = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + 7y_4 + 5y_5 + 2y_6 + 5y_7]$$

is known as Weddles rule as approximate quadrature formula.

Q.2

(a) Prove that divided difference are symmetrical in all their arguments i.e. the value of any divided difference is independent of the order of the argument.

⇒ Proof:

$$f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= \frac{f(x_0)}{x_0 - x_1} - \frac{f(x_1)}{x_0 - x_1}$$

$$= \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0}$$

$$= \frac{f(x_0)}{x_0 - x_1}$$

Thus $f(x_0, x_1)$ is symmetrical in (x_0, x_1)

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}$$

$$= \frac{\frac{f(x_0) - f(x_1)}{x_0 - x_1} - \frac{f(x_1) - f(x_2)}{x_1 - x_2}}{x_0 - x_2}$$

$$= \frac{x_1 f(x_0) - x_1 f(x_1) - x_2 f(x_0) + x_2 f(x_1)}{(x_0 - x_1)(x_1 - x_2)}$$

$$= \frac{-[x_0 \{f(x_1) - f(x_2)\} - x_1 \{f(x_1) - f(x_2)\}]}{(x_0 - x_1)(x_1 - x_2)}$$

$$= \frac{f(x_0)}{x_0 - x_1} - \frac{f(x_2)}{x_1 - x_2} + \frac{f(x_1)(x_2 - x_0)}{(x_0 - x_1)(x_1 - x_2)}$$

$$= \sum \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

Which is a symmetrical in x_0, x_1, x_2

Now, we have to show

$$f(x_0, x_1, x_2, \dots, x_n) = \frac{f(x_0)}{(x_0 - x_1) \dots (x_0 - x_n)}$$

$$+ \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} + \dots + \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}$$

$$+ \dots + \frac{f_0}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}$$

So the coefficient of $f(x_i)$ is $\gamma_i^{(n)}$

$$\gamma_i^{(n)} = \frac{1}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

Where,

$i = 0, 1, 2, \dots, n$ and zero factor

$x_i - x_i$ is omitted in denominator

We assume that result hold good for n
Now, for

$$f(x_0, x_1, \dots, x_{n+1}) = \frac{1}{(x_{n+1} - x_0)} \left\{ f(x_1, x_2, \dots, x_{n+1}) - f(x_0, x_1, \dots, x_n) \right\}$$

$$= \frac{1}{(x_{n+1} - x_0)} \left\{ \frac{1}{(x_i - x_1) \dots (x_i - x_{n+1})} - \frac{1}{(x_i - x_0) \dots (x_i - x_n)} \right\}$$

Here the coefficient of $f(x_i)$ is

$$= \frac{1}{(x_{n+1} - x_0)} \left\{ \frac{1}{(x_i - x_1) \dots (x_i - x_{n+1})} - \frac{1}{(x_i - x_0) \dots (x_i - x_n)} \right\}$$

$$= \frac{1}{(x_i - x_0) (x_i - x_1) \dots (x_i - x_{n+1})}$$

$$= \alpha_i^{(n+1)}$$

$\therefore \alpha_i^{(n+1)}$ is of the same form $\alpha_i^{(n)}$ but has $(n+1)$ factors

$\Rightarrow$ It is valid for $(n+1)$

we have seen ① is true for

$n=1, n=2$

So by induction method ① is true for any positive integer n

Hence proved.

Q.3

- (a) Explain Euler's modified method to solve differential equation of first order $\frac{dy}{dx} = f(x, y)$ with initial condition $y = y_0$ when $x = x_0$.

⇒ Solution :-

The first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

let $y' = f(x, y)$

integrating the first from $(x_i \text{ to } x_{i+1})$ to x_{i+1}

$$\int_{x_i}^{x_{i+1}} \frac{dy}{dx} = \int_{x_i}^{x_{i+1}} y' dx$$

$$[y]_{x_i}^{x_{i+1}} = \int_{x_i}^{x_{i+1}} y' dx$$

$$(y)_{x_{i+1}} - (y)_{x_i} = \int_{x_i}^{x_{i+1}} y' dx$$

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} y' dx$$

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} y' dx$$

with $y'_i = f(x_i, y_i)$

Using the trapezoidal rule for y_{i+1} we get.

$$y_{i+1} = y_i + \frac{1}{2}h[y_i' + y_{i+1}']$$

i.e. $y_{i+1} = y_i + \frac{1}{2}h [f(x_i, y_i) + f(x_{i+1}, y_{i+1})] + R$

where R is error due to approximation in the integral.

⑥ Evaluate

$$I = \int_4^{5.2} \log_e x dx \text{ by using Weddle's rule}$$

x	$\log_e x$
4.0	1.38629463
4.2	1.43508453
4.4	1.48160454
4.6	1.52605030
4.8	1.56861592
5.0	1.60943791
5.2	1.64865863

⇒ Solution :-

Here $h = 4.2 - 4.0 = 0.2$

The Weddle's rule is

$$\int_4^{5.2} \log_e x dx = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{0.6}{10} [1.38629463 + 7.17542265 + 1.681604 + 9.1563378 + 1.60943791 + 8.04718955 + 1.64865863]$$

$$= \frac{18.362967102}{10}$$

$$\int_4^{5.2} \log_e x \, dx = 1.8362967102 \quad \text{Approximately.}$$

© Evaluate $\int_0^1 e^x \, dx$ by using Simpson's $\frac{3}{8}$ rule given

$$e^0 = 1, e^1 = 2.72, e^2 = 7.39, e^3 = 20.09, e^4 = 54.60$$

⇒ Solution :-

Q.4

(a) Prove that Newton's divided difference interpolation formula for unequal intervals.

(b) Prove that the Bessel's interpolation formula.

(c) Find y_{35} using Stirling formula Given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 349$, $y_{50} = 243$

⇒ Solution :-

Here $x = 35$, $h = 10$

Choose the origin at 30 year

i.e. $x_0 = 30$

$$u = \frac{x - x_0}{h} = \frac{25 - 30}{10} = \frac{5}{10} = 0.5$$

u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
-1	512	-73	-20	10
0	439	-93	-10	
1	349	-103		
2	243			

The Stirling formula is

$$y_u = y_0 + u \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{u^2}{2!} \Delta^2 y_{-1}$$

$$+ \frac{u(u^2 - 1)}{3!} (\Delta^3 y_{-1} + \Delta^3 y_{\frac{-2}{2}})$$

$$= 439 + 0.5 \left(\frac{-93 - 73}{2} \right) + \frac{(0.05)(-20)}{2}$$

$$= 439 - 415 - 25$$

$$= \boxed{395} \quad [\therefore = 395]$$

(e) State general quadrature formula

⇒ The general quadrature formula is :

$$I = n \left[4y_0 + \frac{u^2}{2} \Delta y_0 + \frac{\frac{u^3}{3} - \frac{u^2}{2}}{2!} \Delta^2 y_0 + \frac{\frac{u^4}{4} - \frac{3u^3}{3} - \frac{2u^2}{2}}{3!} \Delta^3 y_0 \right]$$

$$\int_0^n y(x) dx$$

$$I = n \left[ny_0 + \frac{n^2}{2} \Delta y_0 + \frac{\frac{n^3}{3} - \frac{n^2}{2}}{2!} \Delta^2 y_0 + \frac{\frac{n^4}{4} - \frac{n^3}{3} + \frac{n^2}{2}}{3!} \Delta^3 y_0 \right]$$

(up to $n+1$ term)

(f) State Simpson's $\frac{1}{3}$ rd rule.

⇒ The Simpson's $\frac{1}{3}$ rd rule is

$$\int_{x_0}^{x_0+nh} y dx = \frac{n}{3} \left[(y_0 + y_n) + 2(y_2 + y_4 + \dots + y_{n-2}) + 4(y_1 + y_3 + \dots + y_{n-1}) \right]$$

Q.2@ Prove that Newton Gregory formula for backward interpolation using polynomial in x of degree n

$$P_n(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-b) + A_3(x-a)(x-a-b)(x-a-2b) + \dots + A_n(x-a)(x-a-b)\dots(x-a-(n-1)b)$$

⇒ Proof:-

Let,

$y = f(x)$ denote a function which takes value $f(a), f(a+b), f(a+2b), \dots$

for $(n+1)$ equidistance values $a, a+b, a+2b, \dots, a+nb$ of the independent variable x .

Let $P_n(x)$ be a polynomial in x degree n

$$P_n(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-b) + A_3(x-a)(x-a-b)(x-a-2b) + \dots + A_n(x-a)(x-a-b)\dots(x-a-(n-1)b)$$

also by 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 788, 789, 790, 791, 792, 793, 794, 795, 796, 797, 798, 799, 800, 801, 802, 803, 804, 805, 806, 807, 808, 809, 810, 811, 812, 813, 814, 815, 816, 817, 818, 819, 820, 821, 822, 823, 824, 825, 826, 827, 828, 829, 830, 831, 832, 833, 834, 835, 836, 837, 838, 839, 840, 841, 842, 843, 844, 845, 846, 847, 848, 849, 850, 851, 852, 853, 854, 855, 856, 857, 858, 859, 860, 861, 862, 863, 864, 865, 866, 867, 868, 869, 870, 871, 872, 873, 874, 875, 876, 877, 878, 879, 880, 881, 882, 883, 884, 885, 886, 887, 888, 889, 890, 891, 892, 893, 894, 895, 896, 897, 898, 899, 900, 901, 902, 903, 904, 905, 906, 907, 908, 909, 910, 911, 912, 913, 914, 915, 916, 917, 918, 919, 920, 921, 922, 923, 924, 925, 926, 927, 928, 929, 930, 931, 932, 933, 934, 935, 936, 937, 938, 939, 940, 941, 942, 943, 944, 945, 946, 947, 948, 949, 950, 951, 952, 953, 954, 955, 956, 957, 958, 959, 960, 961, 962, 963, 964, 965, 966, 967, 968, 969, 970, 971, 972, 973, 974, 975, 976, 977, 978, 979, 980, 981, 982, 983, 984, 985, 986, 987, 988, 989, 990, 991, 992, 993, 994, 995, 996, 997, 998, 999, 1000

We choose Coefficients $A_0, A_1, A_2, \dots, A_n$ so that,

$$P_n(a) = f(a), \quad P_n(a+b) = f(a+b)$$

$$P_n(a+2b) = f(a+2b), \quad \dots, \quad P_n(a+nb) = f(a+nb)$$

Putting this in ① we get

$x = a, a+b, a+2b, \dots, a+nb$ & above values of $P_n(a), P_n(a+b), P_n(a+2b), \dots, P_n(a+nb)$ we get

$$f(a) = A_0, \quad f(a+b) = A_0 + A_1(b-a)$$

$$hA_1 = f(a+b) - A_0$$

$$A_1 = \frac{1}{h} [f(a+h) - f(a)]$$

$$A_1 = \frac{1}{h} \Delta f(a)$$

$$f(a+2h) = A_0 + (a+2h-a)A_1 + A_2(a+2h-a)(a+2h-a-h)$$

$$2h^2 A_2 = f(a+2h) - A_0 - 2hA_1$$

$$= f(a+2h) - f(a) - 2h \frac{1}{h} \Delta f(a)$$

$$= f(a+2h) - f(a) - 2f(a+h) + 2f(a)$$

$$= f(a+2h) - f(a) - 2f(a+h) + 2f(a)$$

$$= f(a+2h) - 2f(a+h) + f(a)$$

$$A_2 = \frac{1}{2!h^2} \Delta^2 f(a)$$

Similarly,

$$A_3 = \frac{1}{3!h^3} \Delta^3 f(a)$$

$$A_4 = \frac{1}{4!h^4} \Delta^4 f(a)$$

$$A_n = \frac{1}{n!h^n} \Delta^n f(a)$$

Substituting this value of equation (1) $A_0, A_1, A_2, \dots, A_n$ we get.

$$P_n(x) = f(a) + \frac{1}{n} \Delta f(a)(x-a) + \dots$$

$$+ \frac{1}{2! h^2} \Delta^2 f(a)(x-a)(x-a-h) + \dots$$

$$+ \frac{1}{3! h^3} \Delta^3 f(a)(x-a)(x-a-h)(x-a-2h) + \dots$$

$$+ \frac{1}{n! h^n} \Delta^n f(a)(x-a)(x-a-h) \dots (x-a-(n-1)h)$$

This is Gregory formula for forward interpolation

backward interpolation - we have

$$f(a) = f(a) - f(a-h)$$

$$f(a-h) = f(a) - \nabla f(a)$$

$$\begin{aligned} f(a-2h) &= (1-\nabla) f(a-h) \\ &= (1-\nabla)(1-\nabla) f(a) \\ &= (1-\nabla)^2 f(a) \end{aligned}$$

Proceeding in this way we get

$$f(a-nh) = (1-\nabla)^n f(a)$$

$$= \left[1 - n\nabla + \frac{n(n-1)}{2!} \nabla^2 - \frac{n(n-1)(n-2)}{3!} \nabla^3 + \dots \right] f(a)$$

$$= f(a) - n\nabla f(a) + \frac{n(n-1)}{2!} f(a) - \frac{n(n-1)(n-2)}{3!} \nabla^3 f(a) + \dots$$

$\therefore$ This is Newton's backward difference formula for backward difference.

(4) Prove that

$$(i) \Delta(U_n V_n) = U_{n+1} \Delta V_n + V_n \Delta U_n$$

$$(ii) \Delta(U_n V_n) = U_n \Delta V_{n+1} + V_{n+1} \Delta U_n$$

$\Rightarrow$ Solution:-

Let u_x & v_x be two function of x then by definition of difference operator

$$\Delta u_x = u_{x+1} - u_x \quad \text{and}$$

$$\Delta v_x = v_{x+1} - v_x$$

where L is the interval of differentiating

(i) Consider

$$L.H.S = \Delta(U_n V_n)$$

$$= u_{n+1} v_{n+1} - u_n v_n$$

$$= u_{n+1} v_{n+1} - u_{n+1} v_n + u_{n+1} v_n - u_n v_n$$

$$= u_{n+1} [v_{n+1} - v_n] + v_n [u_{n+1} - u_n]$$

$$= u_{n+1} \Delta v_n + v_n \Delta u_n$$

$$= R.H.S$$

Hence,

$$\Delta(U_n V_n) = u_{n+1} \Delta v_n + v_n \Delta u_n$$

(ii) Consider,

$$L.H.S = \Delta(U_n V_n)$$

$$= u_{n+1} v_{n+1} - u_n v_n$$

$$= u_{n+1} v_{n+1} - u_n v_{n+1} + u_n v_{n+1} - u_n v_n$$

$$= V_{x+1} [U_{x+1} - U_x] + U_x [V_{x+1} - V_x]$$

$$= V_{x+1} \Delta U_x + U_x \Delta V_x$$

$$= U_x \Delta V_x + V_{x+1} \Delta U_x$$

$$= R.H.S$$

- (a) Prove that the n th divided difference can be expressed as the quotient of two determinants each of order $n+1$.

$\Rightarrow$ Proof :-

The 3rd divided difference $f(x_0, x_1, x_2, x_3)$

$$f(x_0, x_1, x_2, x_3) = \frac{f(x)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$= \frac{f(x_0) \text{ (diff. products of } x_1, x_2, x_3 \text{)}}{\text{diff. products of } x_0, x_1, x_2, x_3}$$

$\therefore$ By important theorem of determinant by variable Vandermonde.

difference product of x_1, x_2, x_3

$$= \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \sum f(x_0) \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

proceeding in this way we can express the higher order difference as a quotient of 2 determinants each of order one more than the order of the difference

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
654	2.8156	$\frac{2.8182 - 2.8156}{658 - 654}$	$\frac{66.0007 - 0.0065}{659 - 654}$	$\frac{0.0066 - 0.00}{661 - 654}$
		$= 0.00065$	$= 0.00001$	
658	2.8182	$\frac{2.8189 - 2.8182}{659 - 658}$	$\frac{0.00065 - 0.0007}{661 - 658}$	$= 0.0000038$
		$= 0.007$		
659	2.8189	$\frac{2.8202 - 2.8189}{661 - 659}$	$= 0.000016$	
		$= 0.00045$		
601				

From the table $x_0 = 654$, $x_1 = 658$, $x_2 = 659$, $x_3 = 601$

$$f(x_0) = 2.8156$$

$$f(x_0, x_1) = 0.00065$$

$$f(x_0, x_1, x_2) = 0.00001$$

$$f(x_0, x_1, x_2, x_3) = 0.0000038$$

∴ By newton's divided difference is

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + \frac{(x-x_0)(x-x_1)}{f(x_0, x_1, x_2)} f(x_0, x_1, x_2, x_3)$$

$$= 2.8156 + (656-654) (0.00065) + \frac{(656-654)(656-658)}{(0.00001)} + \frac{(656-654)(656-658)(656-659)}{(0.0000038)}$$

$$= 2.8156 + 0.0013 + (2)(-2) (0.00001) + (2)(-2)(-3) (0.0000038)$$

$$= 2.8156 + 0.0013 - 0.00004 + 12 \times 0.0000038$$

$$= 2.8156 + 0.0013 - 0.00004 + 0.0000456$$

$$= 2.8169$$

Q Prove that trapezoidal rule is approximate quadrature formula.

⇒ Proof:

The general quadrature formula is

$$I = n \left[n y_0 + \frac{n^2}{2} \Delta y_0 + \frac{\frac{n^3}{3} - \frac{n^3}{2}}{2!} n^2 y_0 + \frac{\frac{n^4}{4} - n^3 + n^2}{3!} \right.$$

$\Delta^3 y_0 + \dots \text{upto } n+1 \text{ term} \left. \right]$

Put $n=1$ we get and neglect second and higher difference.

$$\int_{x_0}^{x_0+n} y dx = \frac{1}{n} \left[y_0 + \frac{1}{2} \Delta y_0 \right]$$

$$= \frac{1}{n} \left[y_0 + \frac{1}{2} (y_1 - y_0) \right]$$

$$= \frac{1}{n} \left[y_0 + \frac{1}{2} y_1 - \frac{1}{2} y_0 \right]$$

$$= \frac{1}{n} \left[\frac{y_0 + y_1}{2} \right]$$

Similarly $\int_{x_0}^{x_0+2h} y dx = \frac{1}{n} \left[\frac{y_1 + y_2}{2} \right]$

$$\int_{x_0}^{x_0+n} y dx = \frac{1}{n} \left[\frac{y_{n-1} + y_n}{2} \right]$$

Adding all integral we get,

$$\int_{x_0}^{x_0+nh} y dx = \frac{1}{n} \left[\frac{y_0 + y_n}{2} + (y_1 + y_2 + \dots + y_{n-1}) \right]$$

∴ This is trapezoidal rule.

(b) Evaluate $\int_0^{\pi/2} \sin x \, dx$ by using Simpsons $\frac{1}{3}$ rd rule,

Given, $\sin 0 = 0$, $\sin \frac{\pi}{20} = 0.1564$, $\sin \frac{\pi}{10} = 0.3090$

$\sin \frac{3\pi}{20} = 0.4540$, $\sin \frac{\pi}{5} = 0.5878$

$\sin \frac{\pi}{4} = 0.7071$, $\sin \frac{3\pi}{10} = 0.8090$

$\sin \frac{7\pi}{20} = 0.8910$, $\sin \frac{2\pi}{5} = 0.9511$

$\sin \frac{9\pi}{20} = 0.9877$, $\sin \frac{\pi}{2} = 1.0000$

⇒ Solution :-

Here,

$b = \frac{\pi}{2}$, $a = 0$, $n = 10$

∴ $h = \frac{b-a}{n} = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$

We know that, the simpson's $\frac{1}{3}$ rd rule

$$\int y dx = \frac{n}{3} [(y_0 + y_n) + 2(y_2 + y_4 + y_6 + y_8 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9 + y_{11})]$$

$$= \frac{\pi}{60} [(0.0000 + 1.000) + 2(0.3090 + 0.5876 + 0.8040 + 0.9511) + 4(0.1564 + 0 + 0.4540 + 0.7071 + 0.8910 + 0.9877)]$$

$$= \frac{\pi}{60} [1 + 2(2.6519) + 4(3.1962)]$$

$$= \frac{\pi}{60} [1 + 2.5519 + 12.7848]$$

$$= \frac{\pi}{60} [19.0980]$$

$$= \frac{59.8771}{60}$$

$$= 0.9999$$

$$= 1 - 0.9974$$

$$= 0.026$$

difference exact simpson rule value

$$= 1 - 0.9995$$

$$= 0.0005$$

① Given: $\frac{dy}{dx} = \frac{y-x}{y+x} = f(x, y)$

with $y=1$ when $x=0$ find approximately the value of y for $x=0.1$ by Picards method (upto 2nd approximation only).

⇒ Solution:-

Here

$$\frac{dy}{dx} = \frac{y-x}{y+x} = f(x, y)$$

Here $y=1$ for $x=0$
 $x_0=0$ $y_0=1$

for this, $\frac{dy}{dx} = \frac{1-0}{1+0} = \frac{1}{1} = 1$

Since $y=1$ and $\frac{dy}{dx} = 1$ at $x=0$

y will be early one for small x so,

we consider the first approximation $u_1(x)=1$

so the second approximation by using Picards method

$$u_2(x) = y_0 + \int_{x_0}^x f(x, u_1(x)) dx$$

$$= 1 + \int_{x_0}^x f(x, y) dx$$

$$= 1 + \int_{x_0}^x \frac{1-x}{1+x} dx$$

$$= 1 + \int_{x_0}^x \left(\frac{2}{1+x} - 1 \right) dx$$

$$= 1 + [2 \log(1+x) - x]_0^x$$

$$= 1 + [2 \log(1+x) - x(0.0)]$$

$$= x + 2 \log(1+x) - x$$

for $x=0.1$, $u_2(x) = 1 + 2 \log(1+0.1) - 0.1$

$$= 1 + 2 \log 1.1 - 0.1$$

$$= 1 + 2(0.04139) - 0.1$$

~~$$= 1 + 2(0.04139) - 0.1$$~~

$$= 1 + 0.08278 - 0.1$$

$$= 1.0827 - 0.1$$

$$u_2(x) = 0.98278$$