Numerical Analysis

Q.1 Attempt any five of the following.

(a) Define Argument and Entry.

⇒ Argument :-

If $y = f(x)$ be a function then the values of independent variable x are called Arguments.

denoted by $x_0; x_1, x_2 \ldots x_n$.

Entry :-

The values of dependent variable y corresponding to the argument is called entry.

denoted by $y_0, y_1, y_2 \ldots y_n$.

(b) Evaluate : $\Delta^2 x^3$.

⇒

(D) Define numerical quadrature :-

Numerical quadrature :-

We observe integrals which cannot be easily or exactly evaluated. If the values of integral for few values of variables are given, we cannot integrate. Then the numerical method or integration are used.

Defination :-

The process of finding and evaluating a definite integral from a set of numerical values of integrant, when applied to integration of function is known as quadrature.

(F) State trapezoidal rules as approximate quadrature formula

⇒ Trapezoidal Rule :-

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{1}{4}\left[\frac{y_0+y_n}{2} + (y_1+y_2+y \cdots y_{(n-1)}\right]$$

Called as trapezoidal Rule.

Q.2 Attempt any two of the following.

(a) Prove that n^{th} divided differences of a polynomial of the n^{th} degree are constant.

⇒ Proof :-

The rational integral function (Polynomial) of n^{th} degree in x be

$$\therefore f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l$$

where,

n - is a positive integer $a, b, c, \ldots, k, l$ are constant and $a \neq 0$

$$f(x+h) = a(x+h)^n + b(x+h)^{n-1} + c(x+h)^{n-2} + \cdots + kx + l$$

Now,

$$\Delta f(x) = f(x+h) - f(x)$$
$$= [a(x+h)^n + b(x+h)^{n-1} + \cdots + kx + h + l]$$
$$- [ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l]$$

$$\therefore \Delta f(x) = a_n bx^{n-1} + b'x^{n-2} + c'x^{n-3} + \cdots + k'x + l' \quad \ldots (1)$$

Where,

b', c', k', l' are constant thus the 1^{st} difference of rational integral function (Polynomial) is polynomial of degree $n-1$.

So,

Second divided difference of $f(x)$ is

$$\Delta^2 f(x) = \Delta(\Delta f(x))$$
$$= \Delta(f(x+h) - f(x)]$$
$$= \Delta f(x+h) - \Delta f(x)$$

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$$= a_n h(x+h)^{n-1} + b'(x+h)^{n-2} + c'(x+h)^{n-3} + \cdots + k(x+h) + l)$$
$$- (a_n h x^{n-1} + b' x^{n-2} + c' x^{n-3} + \cdots + k' x + l')$$

$$= a_n(n-1) h^2 x^{n-2} + b'' x^{n-3} + c'' x^{n-4} + \cdots + k'' x + l'$$

The second difference of the n^{th} degree polynomial is a polynomial of degree $n-2$.
By continuing in this manner we arrive at the result

$$\Delta^n f(x) = a[n(n-1)(n-2) \cdots 2 \cdot 1] h^n$$
$$= a h^n n!$$

$$\Delta^n f(x) = a[n(n-1)(n-2) \cdots 2 \cdot 1] h^n$$
$$= a h^n n!$$

So the n^{th} difference is a constant

Hence proved.

(b) Evaluate $\Delta^n(e^{ax+b})$. Where a & b are constants and interval of differentiating is unity.

$\Rightarrow$

We know that,

$\Delta f(x) = f(x+h) - f(x)$

$\therefore \Delta e^{ax+b} = e^{a(x+1)+b} - e^{ax+b}$

$= e^{ax+a+b} - e^{ax+b}$

$= e^{ax+b+a} - e^{ax+b}$

$= e^{ax+b}(e^a - 1)$

Similarly,

$\Delta^2 e^{ax+b} = \Delta[\Delta e^{ax+b}]$

$= \Delta[e^{ax+b}(e^a-1)]$

$= e^{a(x+1)+b}(e^a-1) - e^{ax+b}(e^a-1)$

$= e^{ax+a+b}(e^a-1) - e^{ax+b}(e^a-1)$

$= e^{ax+b+a}(e^a-1) - e^{ax+b}(e^a-1)$

$= (e^{ax+b+a} - e^{ax+b})(e^a-1)$

$= [e^{ax+b}(e^a-1)](e^a-1)$

$= e^{ax+b}(e^a-1)^2$

~~Similarly~~

~~Δ^3~~

Similarly,

$$\Delta^3 e^{ax+b} = e^{ax+b}(e^a-1)^3$$

Continuing this process we get

$$\Delta^n e^{ax+b} = e^{ax+b}(e^a-1)^n$$

(c) From the following table, find the number of students who obtained less than 45 marks-

Marks	Number of Students
30-40	31
40-50	42
50-60	51
60-70	35
70-80	31

⇒ Solution:-

mark less than (a)	No. of students	Δ	Δ^2	Δ^3	Δ^4
40	31	42	9	25	37
50	73	31	16	12	
60	124	35	4		
70	159	31			
80	190				

Here, $h=10$, $x=45$, $a=40$

$$u = \frac{x-a}{h} = \frac{45-40}{10} = \frac{5}{10} = 0.5$$

∴ By newton Forward interpolation formula.

$$P_n(x) = f(a) + u\Delta f(a) + \frac{u(u-1)\Delta^2 f(a)}{2!} + \frac{u(u-1)(u-2)}{3!}$$

$$\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}$$

$$P_n(x) = 40 + (0.5)31 + (0.5)(0.5-1)(-4) + \cancel{60}.$$

$$\frac{(0.5)(0.5-1) \times (0.5-2) \times 12}{3 \times 2} + (0.5)$$

$$\frac{(0.5)(0.5-1)(0.5-2)(0.5-3) \times 37}{4 \times 3 \times 2}$$

$$P_n(x) = 40 + 15.5 + 0.5 + 0.75 - 1.44$$

$$P_n(x) = 56.75 - 1.44$$

$$\boxed{\therefore P_n(x) = 55.31}$$

Q.3

a) Prove that newtons formula for unequal intervals.

⇒ Proof :-

Let $f(x_0), f(x_1), f(x_2) ---- f(x_n)$ be the values of $f(x)$ Corresponding to the arguments.

$x_0, x_1, x_2 --- x_n$ not equally spaced

∴ By the definition of divided difference

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad --- (1)$$

Now

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{} \quad --- (I)'$$

$$\therefore f(x, x_0) = f(x_0 - x_1) + (x - x_1) f(x, x_0, x_1) \quad --- (II)''$$

Similarly,

$$f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2) \quad ---- (I)'''$$

$$f(x, x_0, x_1, x_2) = f(x_0, x_1, x_2, x_3) + (x.x_3) f(x_g, x_0, x_1, x_2, x_3) \quad ---- (1)$$

Similarly,

$$f(x, x_0, x_1, x_2 ---- x_{n-1}) = f(x_0, x_1, x_2 --- x_n) + (x - x_n) f(x, x_0, x_1 ---- x_n) \quad (I)^n$$

Multiply I' by $x - x_0$

I'' by $(x-x_0)(x-x_1)$ and

I''' by $(x-x_0)(x-x_1)(x-x_2)$ and

I^n by $(x-x_0)(x-x_1)\ldots(x-x_n)$ and

add it I we get,

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + \ldots + (x-x_0)(x-x_1)\ldots(x-x_n) f(x_0, x_1, \ldots x_n) +$$

where

$$R_n = (x-x_0)(x-x_1)\ldots(x-x_n) f(x, x_0, x_1, \ldots x_n)$$

Assuming that $f(x)$ is polynomial of n

so,

divided difference $f(x, x_0, x_1, \ldots x_n)$ vanish is zero

Hence,

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) + \ldots\ldots + (x-x_0)(x-x_1)\ldots(x-x_n) f(x_0, x_1, \ldots x_n)$$

This is called newtons divided difference interpolation formula for unequal interval.

(b) Prove Bessel's interpolation formula.

⇒ Proof :-

we know that, the gauss forward formula is

$$y_u = y_0 C_1 \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_{-1} + \frac{u(u^2-1)}{3!} \Delta^3 y_{-1}$$

$$+ \frac{u(u^2-1)(u^3-2)}{4!} \Delta^4 y_{-2} + \ldots\ldots$$

The gauss backward forwaromula with origin transfer to unity takes the form.

$$y_u = y_1 + {}^{u+1}C_1 \Delta y_0 + {}^{u}C_2 \Delta^2 y_0 + {}^{u}C_3 \Delta^3 y_{-1} + \ldots$$

Taking the mean of this, with gaus's forward formula is

$$y_u = \frac{1}{2}(y_0 + y_1) + \left(u - \frac{1}{2}\right) \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_{-1}$$

(c) Interpolate by means of gauss backward formula the population for the year 1936. Given the following table

Years	Population (1000)
1901	12
1911	15
1921	20
1931	27
1941	39
1951	52

⇒ Solution :-

Let $u = \frac{x - x_0}{h}$

here, $x = 1936$, $h = 10$

Take the origin at 1931

$x = \frac{1936 - 1931}{10} = 0.5$

The difference table is

u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
-3	12	3	2	0	3	-10
-2	15	5	2	3	-7	
-1	20	7	5	-4		
0	27	12	1			
1	39	13				
2	52					

The gauss

The gauss Backward formula is

$$y_u = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{u(u^2-1)(u-2)}{4!}\Delta^4 y_{-3} + \frac{u(u^2-1)(u^2-2)}{5!}\Delta^5 y_{-4}$$

$$= 27 + 0.5\times7 + \frac{0.5\times1.5\times2}{1\times2} + \frac{0.5\times0.75\times3}{1\times2\times3} + \frac{(0.5)(-0.75)(2.5)\times3}{1\times2\times3\times4}$$

$$+ \frac{(0.5)(-0.75)(-3.75)(-10)}{1\times2\times3\times4\times5}$$

$= 27 + 3.5 + 0.75 - 0.1875 + 0.11718 - 0.11718$
$= 31.25 - 0.1875$
$= 31.0625$

Q4

(a) Prove the general quadrature formula for equidistant ordinates.

$\Rightarrow$ proof :-

Let

$$I = \int_0^b y\,dx \qquad \text{where } , y = f(x)$$

Suppose,

$f(x)$ is a given for certain equidistant values of x Say $x_0 . x_0+h , x_0+2h , \dots x_0+nh$

let,

The range (a,b) be divided into n equal part, each of width h so that,

$\therefore b-a = nh$

let, $x_0 = a$, $x_1 = a+h$ $x_2 = a+2h$
$x_n = a+nh = b$

So,
we assume that n+1 ordinates $y_0, y_1, y_2 \cdots y_n$ are equidistant
So,

$$I = \int_a^b y.dx$$

$$= \int_{x_0}^{x_0+nh} y_x.dx$$

$$= \int_0^n y_{x_0} + hu . hdu$$

where,

$$u = \frac{x - x_0}{h} \Rightarrow dx = hdu$$

using newtons backward formula,

$$y = y_{x_0} + nhu$$

$$P_n(x) = P_n(x_0 + nu)$$

$$I = n\int_0^n [y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots] du$$

$$= n[y_0 u + \frac{u^2}{2}\Delta y_0 + \frac{\frac{u^3}{3} - \frac{u^2}{2}}{2!} + \frac{\frac{u^4}{4} - \frac{3u^3}{3} - 3}{3!} + \Delta^3 y_0$$

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{\frac{n^3}{3} - \frac{n^2}{2}}{2!}\Delta^2 y_0 + \frac{n^4 - n^3 + n^3}{3!}\Delta^3 y_0 \right.$$

+ upto n+1 term

This is general quadrature formula.

(b) Apply Simpson's rule to estimate the value of the integral : $\int_1^2 \frac{dx}{x}$

By dividing the intervals (1,2) into 4 equal parts.

⇒ Solutions :-

Here,

$\therefore y = f(x) = \frac{1}{(x)}$

$a = 1$, $b = 2$, $h = 4$

$h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$

x : $1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$

y ; $1, \frac{4}{5}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}$

By Simpson's $\frac{1}{3}$rd rulue is

$$\int_a^b y\,dx = \frac{1}{3} \times h[(y_0 + y_n) + 2(y_2 + y_4 + \dots + y_{n-2}) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1})]$$

$$\int_1^2 \frac{dx}{x} = \frac{1}{4/3}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{2}{3}\right)+4\left(\frac{4}{5}+\frac{4}{7}\right)\right.$$

$$= \frac{1}{12}\left[\frac{3}{2}+\frac{4}{3}+5.4857142856\right]$$

$$= \frac{1}{12}\left[2.833333 + 5.485142850\right]$$

$$= \frac{1}{12} \times 8.3190476189$$

$$= 0.69325395$$

$$= 0.77 \text{ Approxiametly}$$

By simpson's $\frac{3^{th}}{8}$ rule

$$\int_a^b y\,dx = \frac{3}{8} \times h \; [(y_0+y_n) + 3(y_1+y_2+y_4+y_5+y_7+y_8 - - - - +$$
$$y_{n-2} + y_{n-1}) + 2(y_3+y_6+y_9+ - - - - + y_{n-3})$$

$$\int_1^2 \frac{dx}{x} = \frac{3}{8} + \frac{1}{9}\,[(y_0+y_n) + 3(y_1+y_n) + 2(y)$$

$$= 0.09375\left[\left(1+\frac{1}{2}\right)+3\left(\frac{4}{5},+\frac{2}{3}\right) + 2\left(\right.\right.$$

$$= \cancel{0.00}$$

$= 0.09375 \times 7.0428571429$

$= 0.660267$

ⓒ Given :

$\frac{dy}{dx} = \frac{y-x}{y+x}$ with the boundary condition $y=1$ for $x=0$

find approximately for $x=0.1$ by euler's method (two steps)

⇒ Solution :-

We have initial condition

$y=1$ for $x=0$

$x_0=0$, $y_0=1$ and $n=5$ $x=0.1$

$\Delta x = h = \frac{x-x_0}{n} = \frac{0.1}{5} = 0.02$

Step I :- By euler's method

$y_1 = y_0 + f(x_0, y_0)$

Here,

$\frac{dy}{dx} = \frac{y-x}{y+x} = f(x,y)$

$f(x_0, y_0) = \frac{y_0 - x_0}{y_0 + x_0} = \frac{1-0}{1+1} = 1$

$y_1 = 1 \times (0.2 \times 1)$

$y_1 = 1.02$

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Step II :- $y_2 = y_1 + h f(x_1, y_1)$

Here

$y_1 = 1.02$, $x_1 = x_0 + h = 0 + 0.02 = 0.02$

$f(x_1, y_1) = \frac{y_1 - x_1}{y_1 + x_1} = \frac{1.02 - 0.02}{1.00 + 0.02} = \frac{1}{1.04}$

$= 0.96153$

$\therefore y_2 = y_1 + h f(x_1, y_1)$

Step III :-

By euler's method

$y_3 = y_2 + h f(x_2, y_2)$

Here,

$y_2 = 1.0392$, $x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

$f(x_2, y_2) = \frac{y_2 - x_1}{y_2 - x_1} = \frac{1.0392 - 0.04}{1.0392 - 0.04}$

$= \frac{0.9992}{1.07923}$

$= 0.92584$

$y_3 = y_2 + h f(x_2 \cdot y_2)$

$= 1.0392 + (0.02)(0.92584)$

$y_3 = 1.0577$

Step IV :- By euler's method.

Here, $y_4 = y_3 + h f(x_3, y_3)$

$y_3 = 1.0577$, $x_3 = x_2 + h = 0.04 + 0.02$

$= 0.06$

$$f(x_3, y_3) = \frac{y_3 - x_3}{y_3 + x_3} = \frac{1.0577 - 0.06}{1.0577 + 0.06}$$

$$= \frac{0.9977}{1.1177}$$

$$= 0.8926$$

M - 2018

Q.1

(a) Prove that $E\Delta = \Delta E$

$\Rightarrow$ Proof :-

$$E\Delta f(x) = E(f(x+h) - f(x))$$
$$= Ef(x+h) - Ef(x)$$
$$= f(x+2h) - f(x+h)$$
$$= \Delta f(x+h)$$
$$= \Delta E f(x)$$

$\therefore E\Delta = \Delta E$

(b) Define third divided difference of f(x) for arguments x_0, x_1, x_2, x_3

$\Rightarrow$

$$\underset{x_1 x_2 x_3}{\Delta^{3}} f(x_0) = \underset{x_1 x_2 x_3}{\Delta^{1}} f(x_0) = f(x_0, x_1, x_2, x_3)$$
$$= \frac{f(x_0, x_1, x_2) - (f(x_1, x_2, x_3))}{x_0 - x_3}$$

(c) Prove that $\delta = \Delta E^{1/2}$

$\Rightarrow$ Proof :-

(d) Define difference product of x_1, x_2, x_3 in term of determinants.

$\Rightarrow$

Date

(e) Define Numerical Quadrature
⇒

(f) State weddles rule as approximate quadrature formula.
⇒ The general quadrature formula is

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{\frac{n^3}{3} - \frac{n^2}{2}}{2!}\Delta^2 y_0 + \right.$$

$$\left(\frac{n^4}{4} + n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} +$$

$$\left.\left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11}{3}n^3 - 3n^2\right)\frac{\Delta^4 y_0}{4!}\right.$$

Then

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7\right]$$

is known as weddles rule as approximate quadrature formula.

Q.2

ⓐ Prove that divided difference are symmetrical in all their arguments i.e. the value of any divided difference is independent of the order of the argument.

⇒ Proof:

$$f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= \frac{f(x_0)}{x_0 - x_1} - \frac{f(x_1)}{x_0 - x_1}$$

$$= \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0}$$

$$= \sum \frac{f(x_0)}{x_0 - x_1}$$

Thus $f(x_0, x_1)$ is symmetrical in (x_0, x_1)

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}$$

$$= \frac{\frac{f(x_0) - f(x_1)}{x_0 - x_1} - \frac{f(x_1) - f(x_2)}{x_1 - x_2}}{x_0 - x_2}$$

$$= x_1 f(x_0) - x_1 f(x_1) - x_2 f(x_0) + x_2 f(x_1)$$

$$= \frac{-[x_0\{f(x_1) - f(x_2)\} + x_1\{f(x_1) - f(x_2)\}]}{(x_0 - x_1)(x_1 - x_2)}$$

$$= \frac{f(x_0)}{x_0-x_1} - \frac{f(x_2)}{x_1-x_2} + \frac{f(x_1)(x_2-x_0)}{(x_0-x_1)(x_1-x_2)}$$

$$= \sum \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}$$

Which is a symmetrical fn x_0, x_1, x_2

Now, we have to show

$$f(x_0, x_1, x_2, \ldots x_n) = \frac{f(x_0)}{(x_0-x_1)\ldots(x_0-x_n)}$$

$$+ \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} + \ldots\ldots + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}$$

$$\ldots (x_i - x_i - 1)(x_i - x_i + 1) + \ldots + \frac{f_0}{(x_n-x_0)(x_n-1)\ldots(x_n-x_n-1)}$$

So the coefficient of $f(x_i)$ is $\alpha_i(n)$

$$\alpha_i^{(n)} = \frac{1}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_i-1)(x_i-x_i+1)\ldots(x_i-x_n)}$$

Where,
$i = 0, 1, 2, \ldots n$ and zero factor
$x_i - x_i$ is omited in denominator

We assume that result hold good for n
Now, for

$$f(x_0, x_1 \ldots x_{n+1}) = \frac{1}{(x_{n+1} - x_0)} \left\{ \begin{array}{l} f(x_1, x_2 \ldots x_{n+1}) \\ - f(x_0, x_1, \ldots x_n) \end{array} \right\}$$

$$= \frac{1}{(x_{n+1} - x_0)} \left\{ \frac{1}{(x_i - x_i) \ldots (x_i - x_{n+1})} - \frac{1}{(x_i - x_0) \ldots (x_i - x_n)} \right\}$$

Here the coefficient of $f(x_i)$ is

$$= \frac{1}{(x_{n+1} - x_0)} \left\{ \frac{1}{(x_i - x_1) \ldots (x_i - x_{n+1})} - \frac{1}{(x_i - x_0) \ldots (x_i - x_n)} \right\}$$

$$= \frac{1}{(x_i - x_0)(x_i - x_i) \ldots (x_i - x_{n+1})}$$

$$= \alpha_i^{(n+1)}$$

$\therefore \alpha_i^{(n+1)}$ is of the same form $\alpha_i^{(n)}$ but has $(n+1)$ factors

$\Rightarrow$ It is valid for $(n+1)$
we have seen ① is true for
$n=1$, $n=2$
So by induction method ① is true for any positive integer n

Hence proved.

Q.B

(a) Explain Euler's modified method to solve differential equation of first one $\frac{dy}{dx} = f(x,y)$ with initial condition

$y = y_0$ when $x = x_0$.

⇒ Solution :-

The first order differential equation

$$\frac{dy}{dx} = f(x,y)$$

let :- $y' = f(x,y)$

integrating the first from $(x_i (x_0 + ih)$ to x_{i+1}

$$\int_{x_i}^{x_i+h} \frac{dy}{dx} = \int_{x_i}^{x_{i+1}} y'dx$$

$$[y]_{x_i}^{x_i+h} = \int_{x_i}^{x_{i+1}} y'dx$$

$$(y)x_i+h - (y)x_i = \int_{x_i}^{x_{i+1}} y'dx$$

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} y'dx$$

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} y'dx$$

with $y_i' = f(x_i, y_i)$.

using the trapezoidal rule for y_{i+1} we get.

$$y_{i+1} = y_i + \frac{1}{2}h\left[y_i - y_{i+1}\right]$$

i.e.

$$y_{i+1} = y_i + \frac{1}{2}h\left[f(x_i - y_i) + f(x_{i+1}, y_{i+1})\right] + R$$

where R is error due to approximation in the integral.

ⓑ Evaluate

$$I = \int_4^{5.2} \log_e x \, dx$$ by using weddl's rule

x	$\log_e x$
4.0	1.38629463
4.2	1.43508453
4.4	1.84160464
4.6	1.52605630
4.8	1.56861592
5.0	1.60943791
5.2	1.64865863

⇒ Solution :-

Here $h = 4.2 - 4.0 = 0.2$

The weddle's rule is

$$\int_4^{5.2} \log_e x dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_5 + 5y_5 + y_6\right]$$

$$= \frac{0.6}{10}[1.38629463 + 7.17542265 + 1.681604 + 9.1563378 + 1.60943791 + 8.04718955 + 1.64865863]$$

$$= \frac{18.3629677102}{10}$$

$$\int_{4}^{5.2} \log_e x \, dx = 1.8362967102 \text{ Approximately.}$$

ⓒ Evaluate $\int_0^1 e^x \, dx$ by using simpsons $\frac{3}{8}$th rule given

$e = 1$, $e^1 = 2.72$, $e^2 = 7.34$, $e^3 = 20.09$, $e^4 = 54.60$

⇒ solution :-

Q.4

(a) Prove that newtons divided difference interpolation formula for unequal intervals.

(b) Prove that the Bessel's interpolation formula.

(c) Find y_{35} using stirling formula Given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 349$, $y_{50} = 243$

⇒ Solution :-

Here $x = 35$, $h = 10$

Choose the origin at 30 year

i.e. $x_0 = 30$

$$u = \frac{x - x_0}{4} = \frac{25-30}{10} = \frac{5}{10} = 0.5$$

u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
-1	512	-73	-20	10
0	439	-93	-10	
1	346	-103		
2	243			

The stirling formula is

$$y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1}$$

$$+ \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$= 439 + 0.5\left(\frac{-93-73}{2}\right) + \frac{(0.05)(-20)}{2}$$

$$= 43.9 - 415 - 25$$

$\cancel{= 395}$ $\boxed{\therefore = 395}$

(e) State General quadrature formula

⇒ The general Quadrature formula is :

$$I = h\left[uy_0 + \frac{u^2}{2}\Delta y_0 + \frac{\frac{u^3}{3} - \frac{u^2}{2}}{2!}\Delta^2 y_0 + \frac{\frac{u^4}{4} - \frac{3u^3}{3} - \frac{2u^2}{2}}{3!}\Delta^3 y_0 + \ldots \right]_0^n$$

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{\frac{n^3}{3} - \frac{n^2}{2}}{2!}\Delta^2 y_0 + \frac{n^4 - n^3 + n^2}{3!}\Delta^3 y_0 + \ldots + \text{up to } n+1 \text{ term}\right]$$

(f) State Simpson's $\frac{1}{3}$rd rule.

⇒ The Simpson's $\frac{1}{3}$rd rule is

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\begin{bmatrix} (y_0 + y_n) + 2(y_2 + y_4 + \ldots + y_{n-2}) \\ + 4(y_1 + y_3 + y_5 + \ldots\ldots y_{(n-1)}) \end{bmatrix}$$

Q.2 (a) Prove that newton gregory formula for backward interpolation using polynomial in x of degree n

$$P_n(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + A_3(x-a)(x-a-h)(x-a-2h) + \dots + A_n(x-a)(x-a-h) + \dots + (x-a(n-1))$$

⇒ Proof:-

Let,

$y = f(x)$ denote a function which takes value $f(a), f(a+h), f(a+2h) \dots$

for $(n+1)$ equidistance values $a, a+h, a+2h, \dots a+nh$ of the independent variable x.

Let, $P_n(x)$ be a polynomial in x degree n

$$P_n(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + A_3(x-a)(x-a-h)(x-a-2h) + \dots + A_n(x-a)(x-a-h) + \dots + (x-a(n-1)h) \quad \text{--- (1)}$$

we choose coefficient $A_0, A_1, A_2, \dots A_n$ so that,

$P_n(a) = f(a)$, $P_n(a+h) = f(a+h)$

$P_n(a+2h) = f(a+2h) \dots P_n(a+nh) = f(a+nh)$

Putting this in (1) we get

$x = a, a+h, a+2h, \dots a+nh$ & above values of $P_n(a)$, $P_n(a+h)$, $P_n(a+2h) \dots P_n(a+nh)$ we get

$f(a) = A_0$, $f(a+h) = A_0 + A_1(a+h-a)$

$hA_1 = f(a+h) = A_0$

$$A_1 = \frac{1}{h}\left[f(a+h) - f(a)\right]$$

$$A_1 = \frac{1}{h}\Delta f(a)$$

$$f(a+2h) = A_0 + (a+2h-a)A_1 + A_2(a+2h-a)(a+2h-a-h)$$

$$2h^2 A_2 = f(a+2h) - A_0 - 2hA_1$$

$$= f(a+2h) - f(a) - 2h\frac{1}{h}\Delta f(a)$$

$$= f(a+2h) - f(a) - 2f(a+h) + 2f(a)$$

$$= f(a+2h) - f(a) - 2f(a+h) + 2f(a)$$

$$= f(a+2h) - 2f(a+h) + f(a)$$

$$A_2 = \frac{1}{2!h^2}\Delta^2 f(a)$$

Similarly,

$$A_3 = \frac{1}{3!h^3}\Delta^3 f(a)$$

$$A_4 = \frac{1}{4!h^4}\Delta^4 f(a)$$

⋮

$$A_n = \frac{1}{n!h^n}\Delta^n f(a)$$

Substituting this values of equation (1) $A_0, A_1, A_2, \dots A_n$ we get.

$$P_n(x) = f(a) + \frac{1}{h} \Delta f(a)(x-a) +$$

$$\frac{1}{2!\,h^2} \Delta^2 f(a)(x-a)(x-a-h) +$$

$$\frac{1}{3!\,h^3} \Delta^3 f(a)(x-a)(x-a-h)(x-a-2h) +$$

$$---- + \frac{1}{n!\,h^n} f(a)(x-a)(x-a-h) --- (x-a-n-1)$$

This is gregory formula for forward interpolation

backward interpolation: - we have

$f(a) = f(a) - f(a-h)$

$f(a-h) = f(a) - \nabla f(a)$

$$\begin{aligned} f(a-2h) &= (1-\Delta)\, f(a-h) \\ &= (1-\nabla)(1-\nabla) f(a) \\ &= (1-\nabla)^2 f(a) \end{aligned}$$

Proceding in this way we get

$f(a-nh) = (1-\nabla)^n f(a)$

$$= [1 - n\nabla + \frac{n(n-1)\nabla^2}{2!} - \frac{n(n-1)(n-2)\nabla^3}{3!}$$

$$= f(a) - n\nabla f(a) + \frac{n(n-1)}{2!} f(a) - \frac{n(n-1)(n-2)}{3!} \nabla^3 f(a)$$

$$+ ---+$$

$\therefore$ This is newtons backward difference formula for backward difference.

(4) Prove that

(i) $\Delta(U_x V_x) = U_{x+1}\,\Delta V_x + V_x\,\Delta U_x$

(ii) $\Delta(U_x V_x) = U_x\,\Delta V_x + V_{x+1}\,\Delta U_x$

$\Rightarrow$ Solution :-

Let U_x & V_x be two function of x then by defination of difference operator

$\Delta U_x = U_{x+1} - U_x$ and

$\Delta V_x = V_{x+1} - V_x$

where L is the interval of differentiating

(i) Consider

$$\begin{aligned} L.H.S. &= \Delta(U_x V_x) \\ &= U_{x+1}V_{x+1} - U_x V_x \\ &= U_{x+1}V_{x+1} - U_{x+1}V_x + U_{x+1}V_x - U_x V_x \\ &= U_{x+1}[V_{x+1} - V_x] + V_x[U_{x+1} - U_x] \\ &= V_{x+1}\,\Delta V_x + V_x\,\Delta U_x \\ &= R.H.S \end{aligned}$$

Hence,

$\Delta(U_x V_x) = U_{x+1}\,\Delta V_x + V_x\,\Delta U_x$

(ii) Consider,

$$\begin{aligned} L.H.S &= \Delta(U_x V_x) \\ &= U_{x+1}V_{x+1} - U_x V_x \\ &= U_{x+1}V_{x+1} - U_x V_{x+1} + U_x V_{x+1} - U_x V_x. \end{aligned}$$

$= V_{x+1}[U_{x+1} - U_x] + U_x[V_{x+1} - V_x]$

$= V_{x+1}\Delta U_x + U_x \Delta V_x$

$= U_x \Delta V_x + V_{x+1}\Delta U_x$

$=$ R.H.S

(a) Prove that the n^{th} divided difference can be expressed as the quotient of two determinants each of order $n+1$.

$\Rightarrow$ Proof :-

The 3^{rd} divided differences $f(x_0, x_1, x_2, x_3)$

$$f(x_0, x_1, x_2, x_3) = \frac{f(x)}{\sum (x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$= \sum \left[\frac{f(x_0)\,(\text{diff. products of } x_1, x_2, x_3}{\text{diff. products of } x_0, x_1, x_2, x_3)} \right]$$

$\therefore$ By important theorem of determinant by variable Vandemonde.

difference product of x_1, x_2, x_3

$$= \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \sum \left[f(x_0) \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} \right].$$

Proceeding in this way we can express the higher order difference as a quotient of 2 determinants each of order one more than the order of the difference

x	$f(x)$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
654	2.8156	$\frac{2.8182-2.8156}{658-654} = 0.00065$	$\frac{0.0007-0.0065}{659-654} = 0.00001$	$\frac{0.0066-0.00}{661-654} = 0.0000038$
658	2.8182	$\frac{2.8189-2.8182}{659-658} = 0.007$	$\frac{0.00065-0.0007}{661-658} = 0.000016$	
659	2.8189	$\frac{2.8202-2.8189}{661-659} = 0.00045$		
601				

From the table $x_0 = 654$, $x_1 = 658$, $x_2 = 659$, $x_3 = 601$

$f(x_0) = 2.8156$

$f(x_0, x_1) = 0.00065$

$f(x_0, x_1, x_2) = 0.00001$

$f(x_0, x_1, x_2, x_3) = 0.0000038$

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∴ By newtons divided difference is :

$f(x) = f(x_0) + (x-x_0) f(x_0,x_1)\ f(x-x_0)\ (x-x_1)$
$f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)\ f(x_0,x_1,x_2,x_3)$

$= 2.8156 + (656-654)(0.00065) + (656-658)(656-659)$
$(0.00001) + (656-654)(656-658)(656-659)(0.0000038)$

$= 2.8156 + 0.0013 + (2)(-2)(0.00001) + (2)(-2)(-3)(0.0000038)$

$= 2.8156 + 0.0013 - 0.00004 + 12 \times 0.0000038$

$= 2.8156 + 0.0013 - 0.00004 + 0.0000456$

$= 2.8169$

(a) Prove that trapezoidal rule as approximate quadrature formula.

⇒ Proof:

The general Quadrature formula is

$$I = n\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{\frac{n^3}{3} - \frac{n^3}{2}}{2!}\Delta^2 y_0 + \frac{\frac{n^4}{4} - n^3 + n^2}{3!}\Delta^3 y_0 + \dots \text{upto } n+1 \text{ term}\right]$$

Put $n=1$ we get and neglect second and higher difference.

$$\int_{x_0}^{x_0+1h} y\,dx = \frac{1}{h}\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= \frac{1}{h}\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{1}{h}\left[y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0\right]$$

$$= \frac{1}{h}\left[\frac{y_0+y_1}{2}\right]$$

Similarly $$\int_{x_0}^{x_0+2h} y\,dx = \frac{1}{h}\left[\frac{y_1+y_2}{2}\right]$$

$$\vdots$$

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{1}{h}\left[\frac{y_{n-1}+y_n}{2}\right]$$

Adding all integral we get,

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{1}{n}\left[\frac{y_0+y_n}{2} + (y_1+y_2+ \cdots y_{n-1})\right.$$

∴ This is trapezoidal rule.

(b) Evaluate : $\int_0^{\pi/2} \sin x . dx$ by using Simpsons $\frac{1}{3}$rd rule.

Given, $\sin 0 = 0$, $\sin\frac{\pi}{20} = 0.1564$ $\sin\frac{\pi}{10} = 0.3090$

$\sin\frac{3\pi}{20} = 0.4540$ $\sin\frac{\pi}{5} = 0.5878$

$\sin\frac{\pi}{4} = 0.7071$ $\sin\frac{3\pi}{10} = 0.8090$

$\sin\frac{7\pi}{20} = 0.8910$ $\sin\frac{2\pi}{5} = 0.9511$

$\sin\frac{9\pi}{20} = 0.9877$ $\sin\frac{\pi}{2} = 1.0000$

⇒ Solution :-

Here,

$b = \frac{\pi}{2}$ $a = 0$ $n = 10$

$\therefore n = \frac{b-a}{n} = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$

We know that, the simpson's $\frac{1}{3}$rd rule

$$\int y\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+y_8+y_{10})+4(y_1+y_3+y_5+y_7+y_9+y_{11})\right]$$

$$= \frac{\pi}{60}\left[(0.0000+1.000)+2.(0.3090+0.5876+0.8040+0.9511)+4(0.1564+0+0.4540+0.7071+0.8910+0.9877)\right]$$

$$= \frac{\pi}{60}\left[1+2(2.6519)+4(3.1962)\right]$$

$$= \frac{\pi}{60}\left[1+2.5519+12.7848\right]$$

$$= \frac{\pi}{60}\left[19.0986\right]$$

$$= \frac{59.9771}{60}$$

$$= 0.9999$$

$$\approx 1-0.9974$$

$$\approx 0.026$$

difference exact simpson rule value

$$= 1-0.9995$$

$$= 0.0005$$

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① Given: $\frac{dy}{dx} = \frac{y-x}{y+x} = f(x,y)$

with $y=1$ when $x=0$ find approximately the value of y for $x=0.1$ by Picards method (upto 2nd approximation only)

⇒ Solution:-

Here

$\frac{dy}{dx} = \frac{y-x}{y+x} = f(x,y)$

Here, $y=1$ for $x=0$

$x_0=0$ $y_0=1$

for this, $\frac{dy}{dx} = \frac{1-0}{1+0} = \frac{1}{1} = 1$

Since, $y=1$ and $\frac{dy}{dx} = 1$ at $x=0$

y will be early one for small x so,

we consider the first approximation $u_1(x)=1$

So the second approxiamation by using Picards method

$$u_2(x) = y_0 + \int_{x_0}^{x} f(x, u_1(x))\,dx$$

$$= 1 + \int_{x_0}^{x} f(x,y)\,dx$$

$$= 1 + \int_{x_0}^{x} \frac{1-x}{1+x}\,dx$$

$$= 1 + \int_{x_0}^{x} \left(\frac{2}{1+x} - 1 \right) dx$$

$$= 1 + \left[2\log(1+x) - x \right]_0^x$$

$$= 1 + [2\log(1+x) - x(0.0)]$$

$$= x + 2\log(1+x) - x$$

for $x = 0.1$, $u_2(x) = 1 + 2\log(1+0.1) - 0.1$

$$= 1 + 2\log 1 - 0.1$$

$$= 1 + 2(0.04139) - 0.1$$

$$= 1 + 0.08278 - 0.1$$

$$= 1.0827 - 0.1$$

$$u_2(x) = 0.98278$$