Integral Transform

October - 2016

Page No.	
Date	

Q.1 Attempt any five of the following

(a) Define the laplace transform of the function $f(t)$.

=>

The laplace transform of function $f(t)$ defined for all real number $t \geq 0$ is then function $F(s)$ which is unilateral transform defined by

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

Where

s - is a complex number frequency parameter.

(b) Find the laplace transform $t^2 \cdot \delta(t-4)$

=>

Page No.
Date

(c) Find the inverse laplace transform of $\frac{s+2}{(s+2)^2-25}$

⇒ Solution :- Given that,

$$\frac{s+2}{(s+2)^2-25}$$

To find $L^{-1}\left[\frac{s+2}{(s+2)^2-25}\right]$

By using formula,

$$L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cos at$$

$$\therefore \quad L^{-1}\left[\frac{s+2}{(s+2)^2-25}\right] = \cos 5t$$

d) State the Convolution theorem for inverse laplace transform

⇒

If $L[f_1(t)] = F_1(s)$ and
$L[f_2(t)] = F_2(s)$ then

$$L\left[\int_0^t f_1(x) \cdot f_2(t-x)\,dx\right] = F_1(s) \times F_2(s)$$

This is called as Convolution theorem.

(e) State the shifting property of fourier transform of $f(x)$.

⇒ If $F(s)$ is the Complex Fourier transform of $f(s)$ then

$$F[f(x)-a] = e^{isa} F(s)$$ is called shifting property of fourier transform of $f(x)$.

(f) Define the fourier cosine transform of $f(x)$.

⇒ We know that

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos ux . du . \int_0^\infty f(t) . \cos ut\, dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos sx . ds . \int_0^\infty f(t) . \cos st . dt$$

By putting

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) . \cos st . dt$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(s) . \cos sx . ds$$

∴ $F(s)$ is called fourier cosine transform of $f(x)$.

Page No.
Date

Q.2) Attempt any two of the following

(a) If $L[f(t)] = F(s)$, then prove that $L[t^n . f(t)] = (-1)^n . \frac{d^n}{ds^n}[F(s)]$

⇒ Proof :-
We know that the laplace transform of $f(t)$ is defined as the

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} . f(t)\, dt$$

To prove that
$$L[t^n f(t)] = (-1)^n . \frac{d^n}{ds^n}[F(s)]$$

let,
$$\frac{d}{ds} L[f(t)] = \frac{d}{ds}\int_0^\infty e^{-st} . f(t) . dt$$

∴ By leibnitz Rule.
$$\frac{d}{ds} L[f(t)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st} . f(t)]\, dt$$

$$= \int_0^\infty (-t) . e^{-st} . f(t)\, dt$$

$$= (-1)\int_0^\infty e^{-st} . t . f(t)\, dt$$

$$= (-1)^1 L[t . f(t)]$$

$$\frac{d}{ds}[F(s)] = (-1) L[t . f(t)]$$

$\therefore L[t \cdot f(t)] = (-1)^1 \frac{d}{ds}[F(s)]$

Similarlly, Again differentiate w.r.t. 's'

$$\frac{d^2}{ds^2}[F(s)] = \frac{d}{ds} (-1)^1 L[t \cdot f(t)]$$

$$= (-1)^1 \frac{d}{ds} L[t \cdot f(t)]$$

$$= (-1) \frac{d}{ds} \int_0^\infty e^{-st} \cdot t \cdot f(t) \cdot dt$$

$$= (-1)^1 \int_0^\infty \frac{d}{ds} e^{-st} \cdot t \cdot f(t) dt$$

$$= (-1)^1 \int_0^\infty (-t) \cdot e^{-st} \cdot t \cdot f(t) \cdot dt$$

$$= (-1)^2 \int_0^\infty e^{-st} \cdot t^2 f(t) \cdot dt$$

$$\frac{d^2}{ds^2}[F(s)] = (-1)^2 \int_0^\infty e^{-st} \cdot t^2 f(t) \cdot dt$$

$$= (-1)^2 L[t^2 f(t)]$$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[F(s)]$$

Similarlly

$$L[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3}[F(s)]$$

Continuing this we get

$$\therefore L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Hence proved.

(b) Let $f(t)$ be a periodic function with period T. Then prove that,

$$L[f(t)] = \frac{\int_0^T e^{-st} . f(t) . dt}{1 - e^{-sT}}$$

⇒ Proof :-

Let Laplace transform of function $f(t)$ is defined as

$$L[f(t)] = \int_0^\infty e^{-st} . f(t) . dt$$

$$\therefore L[f(t)] = \int_0^T e^{-st} . f(t) . dt + \int_T^{2T} e^{-st} . f(t) + \int_{2T}^{3T} e^{-st} f(t) . dt + \dots$$

Now, put $t = u + T$ in second integral

$\Rightarrow dt = du$

when $t = T \Rightarrow u = 0$

when $t = 2T \Rightarrow u = T$

Similarly

Put $t = u + 2T$ in third integral

$dt = du$

when $t = 2T \Rightarrow u = 0$

when $t = 3T \Rightarrow u = T$ and so on

$$\therefore L[f(t)] = \int_0^T e^{-st} . f(t) . dt + \int_0^T e^{-s(u+T)} . f(u+T) + \int_0^T e^{-s(u+2T)} . f(u+2T) \, du + \dots$$

$\therefore f(u) = f(u+T) = f(u+2T) = \dots$

$$= \int_0^T e^{-st} . f(t) \, dt + e^{-sT} \int_0^T e^{-st} . f(t) . dt + e^{-2sT} \int_0^T e^{-st} . f(t) \, dt$$

$+ \dots$

Page No.	
Date	

$$= (1 + e^{-sT} + e^{-2sT} + \dots) \int_0^T e^{-st} . f(t) . dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} . f(t) . dt$$

$$\therefore \; L[f(t)] = \int_0^T e^{-st} . f(t) . dt$$

$$\therefore \; L[f(t)] = \frac{\int_0^T e^{-sT} . f(t) . dt}{1-e^{-sT}}$$

c) Find the Laplace transform of $e^{-4t} . \frac{\sin 3t}{t}$.

$\Rightarrow$ Solution :-

Let

$f(t) = \sin 3t$.

$\Rightarrow L[f(t)] = L[\sin 3t] = \frac{3}{s^2+9}$

By using first shifting property

$L[e^{at} . f(t)] = F(s-a)$

$\therefore L[e^{-4t} . \sin 3t] = F(s-(-4)) = F(s+4)$

$\therefore L[e^{-4t} \sin 3t] = \frac{3}{(s+4)^2+9}$

Also we know that,

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s) ds$$

$$= \int_0^\infty \frac{3}{(s+4)^2+9} \cdot ds$$

$$= 3 \int_0^\infty \frac{1}{(s+4)^2+9} \cdot ds$$

$$= 3 \int_0^\infty \frac{1}{(s+4)^2+(3)^2} \, ds$$

$$= 3 \times \frac{1}{3} \tan^{-1} \left(\frac{s+4}{3}\right) \Big]_s^\infty$$

$$= \tan^{-1} \infty - \tan^{-1} \left(\frac{s+4}{3}\right)$$

$$= \frac{\pi}{2} - \tan^{-1} \left(\frac{s+4}{3}\right)$$

$$= \cot^{-1} \left(\frac{s+4}{3}\right)$$

$$\therefore L\left[e^{-4t} \cdot \frac{\sin 3t}{t}\right] = \cot^{-1}\left(\frac{s+4}{3}\right)$$

Q.3

(a) Find the inverse laplace transform of $\frac{s-1}{s^2-6s+25}$

$\Rightarrow$ Solution :-

Let,

$$\frac{s-1}{s^2-6s+25} = \frac{s-3+2}{s^2-6s+9+16}$$

$$= \frac{s-3+2}{(s-3)^2+(4)^2}$$

$$= \frac{s-3}{(s-3)^2+(4)^2} + \frac{2}{(s-3)^2+(4)^2}$$

$$L^{-1}\left[\frac{s-1}{s^2-6s+25}\right] = L^{-1}\left[\frac{s-3}{(s-3)^2+(4)^2}\right] + 2L^{-1}\left[\frac{1}{(s-3)^2+(4)^2}\right]$$

$$= e^{3t}L^{-1}\left[\frac{s}{s^2+(4)^2}\right] + 2e^{3t}L^{-1}\left[\frac{1}{s^2+(4)^2}\right]$$

$$= e^{3t}\cos 4t + 2e^{3t} \times \frac{1}{4}\sin 4t$$

$$= e^{3t}.\cos 4t + e^{3t}\frac{\sin 4t}{2}$$

$$L^{-1}\left[\frac{s-1}{s^2-6s+25}\right] = e^{3t}.\cos 4t + e^{3t}.\frac{\sin 4t}{2}$$

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Page No.
Date

ⓑ Using laplace transforms find the solution of initial value problem $y''+9y=6\cos 3t$
where, $y(0)=2$, $y'(0)=0$

⇒ Solution :-

Taking laplace transform,

$L[y'']+9[L[y]]=6L[\cos 3t]$

$$s^2L[y]-sy(0)-y'(0)+9L[y]=6\left[\frac{s}{s^2+3^2}\right]$$

$$(s^2+9)\,L[y]=2s=\frac{6s}{s^2+3^2}$$

$$(s^2+9)\,L[y]=2s+\frac{6s}{s^2+3^2}$$

$$L[y]=\frac{2s}{(s^2+9)}+\frac{6s}{(s^2+3^2)(s^2+9)}$$

$$=\frac{2s}{s^2+9}+\frac{6s}{(s^2+3^2)^2}$$

$$\therefore\ y=L^{-1}\left[\frac{2s}{s^2+3^2}\right]+L^{-1}\left[\frac{6s}{(s^2+3^2)^2}\right]$$

$$=2L^{-1}\left[\frac{s}{s^2+3^2}\right]+6L^{-1}\left[\frac{s}{(s^2+3^2)^2}\right]$$

$$=2L^{-1}\left[\frac{s}{s^2+3^2}\right]-3L^{-1}\left[\frac{-2s}{(s^2+3^2)^2}\right]$$

Page No.
Date

$$= 2\cos 3t - 3L^{-1}\left[\frac{d}{ds}\left(\frac{1}{s^2+3^2}\right)\right]$$

$$= 2\cos 3t - \cancel{3} \times \frac{1}{\cancel{3}} \times \sin 3t$$

$$= 2\cos 3t - \sin 3t$$

$$y = 2\cos 3t - \sin 3t$$

Q.

Page No.
Date

Q.4
(a) If $F(s)$ is the Complex Fourier transform of $f(x)$, then prove that
$$F\{f(ax)\} = \frac{1}{a}F\left(\frac{s}{a}\right).$$

⇒ Solution :-
We know that,
$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x).dx$$

then
$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} f(ax).dx$$

put $ax = t \Rightarrow a\,dx = dt \Rightarrow dx = \frac{dt}{a}$

When $x = -\infty$, $t = -\infty$
$x = \infty$, $t = \infty$

$$\therefore F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-is(t/a)} f(t) \times \frac{1}{a}\,dt$$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty} e^{-i\left(\frac{s}{a}\right)t}\cdot f(t).dt$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$

∴ Hence proved.

Page No.
Date

(b) Derive the fourier cosine integral.

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos ux\, du \cdot \int_0^\infty f(t).\cos ut\, .dt$$

$\Rightarrow$ Solution :-

We know that,

$$f(x) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(t).\cos u(t-x)\, du dt$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\left[\cos ut . \cos ux + \sin ut . \sin ux\right] du dt$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) . \cos ut . \cos ux . du dt + \frac{1}{J}\int_0^\infty \int_{-\infty}^\infty f(t) . \sin ut \sin ux . du dt$$

$$= \frac{1}{\pi}\int_0^\infty \cos ux . du \int_{-\infty}^\infty f(t). \cos ut . dt +$$

$$\frac{1}{\pi}\int_0^\infty \sin ux . du \int_{-\infty}^\infty f(t) . \sin ut\, . dt$$

Since,

$$\int_0^{2a} f(x)\, dx = \begin{cases} 0 & \text{when } f \text{ is odd} \\ 2\int_0^a f(x).dx & f \text{ is even} \end{cases}$$

~~Case~~

Page No.
Date

Case - I :- When F is odd.

$$\therefore \int_{-\infty}^{\infty} f(t).\cos ut.dt = 0$$

$$\therefore f(x) = \frac{1}{\pi} \int_{0}^{\infty} \sin ux.du \times 2 \int_{0}^{\infty} f(t).\sin ut$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(t).\sin ut.\sin ux.du\,dt$$

Case-II :- When f is even

$$\int_{-\infty}^{\infty} f(t).\sin ut.dt = 0$$

$$\therefore f(x) = \frac{1}{2\pi} \int_{0}^{\infty} \cos ux.du \times 2 \int_{0}^{\infty} f(t).\cos ut.dt$$

$$\therefore f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(t).\cos ut.\cos ux.du.dt$$

$$\therefore f(x) = \frac{2}{\pi} \int_{0}^{\infty} \cos ux.du \int_{0}^{\infty} f(t).\cos ut.dt$$

Hence proved.

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Page No.
Date

ⓒ Find the fourier sine transform of $f(x) = e^{-ax}$.

⇒ Solution:-

We know that, fourier sine transformation is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(s) . \sin 3x \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) . \sin 3t . dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} . \sin 3t \, dt$$

Integral transform - 2016

Page No.
Date

Q.1 Attempt any five of following (2 each)

(a) Find the $L[e^{at}]$

⇒ Given

$f(t) = e^{at}$

Also we know that the laplace transform of function $f(t)$ is defined as

$$L[f(t)] = \int_0^\infty e^{-st} . f(t) . dt$$

$$L[e^{at}] = \int_0^\infty e^{-st} . e^{at} . dt$$

$$= \int_0^\infty e^{-(s-a)t} . dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= \frac{-1}{(s-a)} \left[e^{-(s-a)t} \right]_0^\infty$$

$$= -\frac{1}{(s-a)} \left[e^{-(s-a)\infty} - e^{-(s-a)0} \right]$$

$$= -\frac{1}{(s-a)} [0-1]$$

$$= \frac{1}{s-a}$$

$$\boxed{\therefore L[e^{at}] = \frac{1}{s-a}}$$

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Page No.
Date

(b) Define the unit impulse function.

$\Rightarrow$

(c) Find the inverse laplace transform of $\frac{1}{s^2+25}$

$\Rightarrow$ Solution:-

Let

$$L^{-1}\frac{1}{s^2+25} = \frac{1}{5}\ \frac{5}{s^2+(5)^2} = \frac{1}{5}$$

(d) State the seond shifting property of inverse laplace transform.

$\Rightarrow$

$$L^{-1}[e^{-as}F(s)] = F(t-a)\,u(t-a)$$

Page No.			
Date			

(e) State the change of scale property of Fourier transfo-rm.

⇒ Solution:-

If $F(s)$ is the complex fourier transform $f(x)$ then

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Proof :-

We know that

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isd} f(\theta)\,\frac{dt}{a}$$

(By put $ax = t \Rightarrow dx = \frac{dt}{a}$)

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)}$$

(f) Define the fourier sine transform of $f(x)$.

⇒ The fourier sine transform is as follow

We have,

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \sin ux\,du \int_{0}^{\infty} f(t)\,.\sin ut\,.dt \quad (s = u)$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \sin sx\,ds \int_{0}^{\infty} f(t)\,.\sin st\,.dt$$

By putting $F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) . \sin st . dt$

$\therefore f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(s) . \sin sx . ds$

is called fourier sine transform of $f(x)$.

Q2

(a) Prove that

$L\left[\int_0^t f(t) dt\right] = \frac{1}{s} F(s)$ where $L[f(t)] = F(s)$

=> Proof :-

Let

$L[f(t)] = F(s)$

$= \int_0^\infty e^{-st} . f(t) . dt$ --- (1)

To prove that

$L\left[\int_0^t f(t) . dt\right] = \frac{1}{s} F(s)$

let

$\phi(t) = \int_0^t f(t) dt$

$\phi(0) = 0$ and ---- (2)

$\phi'(t) = f(t)$

As we know

$$L[f'(t)] = s.L[f(t)] - f(0)$$

$$\Rightarrow L[\phi'(t)] = sL[\phi(t)] - \phi(0) -$$

$$= s.L[\int_0^t f(t).dt] - 0$$

$$\therefore \quad L[f(t)] = s.L\left[\int_0^t f(t).dt\right]$$

$$\therefore L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)]$$

$$\therefore \quad L\left[\int_0^t f(t).dt\right] = \frac{1}{s} F(s)$$

∴ Hence proved.

(b) If $L[f(t)] = F(s)$ then prove that,
$L[f(t-a).u(t-a)] = e^{-as} F(s)$

$\Rightarrow$ Proof :-
We know that the laplace transform of function $f(t)$ is defined as

$$L[f(t)] = \int_0^\infty e^{-st} f(t).dt = F(s) \quad \cdots (1)$$

To prove that,

$$L[f(t-a)\,u(t-a)] = e^{-as} F(s)$$

Page No.
Date

We know that the unit step function is defined as

$$u(t-a) = \begin{cases} 0 & ; \ t < a \\ 1 & ; \ t \geq a \end{cases}$$

Therefore

$$L[f(t-a).u(t-a)] = \int_0^\infty e^{-st}.f(t-a).u(t-a)\,dt$$

$$= \int_0^a e^{-st}.f(t-a).u(t-a)\,dt + \int_a^\infty e^{-st}.f(t-a).u(t-a).dt$$

$$= \int_0^a e^{-st}.f(t-a).0\,dt + \int_a^\infty e^{-st}.f(t-a).1.dt$$

$$= 0 + \int_a^\infty e^{-st}.f(t-a)\,dt$$

$$= \int_a^\infty e^{-st}.f(t-a).dt$$

Put $t-a = u$ $\quad dt = du$

$t = u+a$

when $t = a \Rightarrow u = 0$

when $t = \infty \Rightarrow u = \infty$

$$= \int_0^\infty e^{-s(u+a)}\,f(u).du$$

$$= \int_0^\infty e^{-su}.e^{-as}.f(u).du$$

$$= e^{-as} \int_0^\infty e^{-su} . f(u) . du$$

$$= e^{-as} . L[f(u)]$$

$$= e^{-as} . F(s)$$

$$\therefore \; L[f(t-a) . u(t-a)] = e^{-as} F(s)$$

Hence proved.

ⓒ Find the laplace transform of $t\sin at$.

⇒ Solution :-

Let laplace transform of $f(t)$ is defined as

$$L[f(t)] = \int_0^\infty e^{-st} . f(t) . dt$$

$$\therefore \; L[t \sin at] = \int_0^\infty e^{-st} \, t\sin at . dt$$

we known that,

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\therefore \; L[t\sin at] = \int_0^\infty e^{-st} . t . \left[\frac{e^{iat} - e^{-iat}}{2i}\right] dt$$

$$= \frac{1}{2i} \int_0^\infty e^{-st} . t . (e^{iat} - e^{-iat}) . dt$$

$$= \frac{1}{2i}\left\{\int_0^\infty e^{-st}.t.\,e^{iat}\,dt - \int_0^\infty e^{-st}.t\,e^{-iat}.dt\right\}$$

$$= \frac{1}{2i}\left\{L.[e^{iat}.t] - L[e^{-iat}.t]\right\}$$

But, we know that.

$$L[e^{at}.t^n] = \frac{n!}{(s-a)^{n+1}}$$

$$= \frac{1}{2i}\left\{\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2}\right\}$$

$$= \frac{1}{2i}\left\{\frac{\cancel{s^2}+2sia+\cancel{i^2a^2}-\cancel{s^2}+2sia-\cancel{i^2a^2}}{(s-ia)^2\,(s+ia)^2}\right\}$$

$$= \frac{1}{\cancel{2}i}\left\{\frac{\cancel{4}^{2}sia}{(s-ia)(s-ia)(s+ia)(s+ia)}\right\}$$

$$= \frac{2as}{(s^2-i^2a^2)(s^2-i^2a^2)}$$

$$= \frac{2as}{(s^2+a^2)(s^2+a^2)}$$

$$= \frac{2as}{(s^2+a^2)^2}$$

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Page No.
Date

$\therefore L[t\sin at] = \dfrac{2as}{(s^2+a^2)^2}$

Q.3)
(a) Find the inverse laplace transform of $\dfrac{s+4}{s(s-1)(s^2+4)}$

⇒ Solution :-

$$\frac{s+4}{s(s-1)(s^2+4)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{Cs+D}{(s^2+4)}$$

$$= \frac{A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)(s)(s-1)}{s(s-1)(s^2+4)}$$

$$= A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)(s)(s-1) = s+4 \quad \ldots (1)$$

$\therefore A(s^3+4s-s^2-4)+B(s^3+4s) + C(Cs^3 - Cs^2+Ds^2-Ds) = s+4$

$(A+B+C)s^3 + (-A-C+D)s^2 + (4A+4B-D)s + (-4A)$
$= 0s^3+0s^2+s+4$

$A+B+C = 0$
$-A-C+D = 0$
$4A+4B-D = 1$
$-4A = 4$
$A = -1$

Put $s=1$
$5B=5 \Rightarrow B=1$
$\therefore D=-1$
$C=0$

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$$\frac{s+4}{s(s-1)(s^2+4)} = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}$$

$$= L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= 1 + e^{t} - \frac{1}{2}\sin 2t$$

$$L^{-1}\left[\frac{s+4}{s(s-1)(s^2+4)}\right] = 1 + e^{t} - \frac{1}{2}\sin 2t.$$

(b) Applying Convolution theorem, solve the following initial value problem

$$y'' + y = \sin 3t$$

$$y(0) = 0 \; ; \; y'(0) = 0$$

⇒ Solution :-

Given that,

$$y'' + y = \sin 3t$$

Taking Laplace transform

$$L[y''] + L[y] = L[\sin 3t]$$

$$s^2 L[y] - s\,y(0) - y'(0) + L[y] = L[\sin 3t]$$

$$s^2 L[y] - 0 - 0 + L[y] = L[\sin 3t]$$

$$s^2 L[y] + L[y] = L[\sin 3t]$$

Page No.
Date

$$(s^2+1)\,L[Y] = \frac{3}{s^2+9}$$

$$L[Y] = \frac{3}{(s^2+1)(s^2+9)}$$

$$Y = L^{-1}\left[\frac{3}{(s^2+1)\ (s^2+9)}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+1)}\right] = \sin t \ , \ L^{-1}\left[\frac{3}{s^2+9}\right] = \sin 3t$$

$$\cos(t-3(t-x)) = \cos t . \cos 3(t-x) - \sin t \sin 3(t-x)$$

$$\cos(t-3(t+x)) = \cos t . \cos 3(t-x) + \sin t . \sin 3(t-x)$$

$$-2\sin t . \sin 3(t-x) = \cos(t+3)t-3x) - \cos(t-3t+3x)$$

$$\sin t . \sin 3(t-x) = \frac{1}{2}\cos(3x-2t) - \frac{1}{2}\cos(4t-3x)$$

$$y = \frac{1}{2}\int_0^t [\cos(3x-2t - \cos(4t-3x)]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(3x-2t)}{3} - \frac{\sin(4t-3x)}{-3}\right]_0^t$$

$$= \frac{1}{6}\left[\sin(3t-2t) + \sin(4t-3t) - \sin(3t-2t) - \sin(4t-3.0)\right]$$

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$$= \frac{1}{6}\left[\sin t + \sin t + \sin 2t - \sin 4t\right]$$

$$y = \frac{1}{6}\left[2\sin t + \sin 2t - \sin 4t\right]$$

(c) Find the inverse laplace transform of

$$\frac{2s-5}{9s^2-25}$$

⇒ Solution :-

Let

$$L^{-1}\left[\frac{2s-5}{9s^2-25}\right] = L^{-1}\left[\frac{2s}{9s^2-25} - \frac{5}{9s^2-25}\right]$$

$$= \frac{2}{9} L^{-1}\left[\frac{s}{s^2-\frac{25}{9}}\right] - \frac{5}{9} L^{-1}\left[\frac{1}{s^2-\frac{25}{9}}\right]$$

$$= \frac{2}{9}\cosh\frac{5}{3} - \frac{5}{9} \times \frac{1}{5/3}\sinh\frac{5}{3}t$$

$$= \frac{2}{9}\cosh\frac{5}{3} - \frac{\cancel{5}}{\cancel{9}} \times \frac{\cancel{3}}{\cancel{5}}\sinh\frac{5}{3}t$$

$$= \frac{2}{9}\cosh\frac{5}{3} - \frac{1}{3}\sinh\frac{5}{3}t$$

Page No.
Date

Q.4

(a) If $F_1(s)$ and $F_2(s)$ are fourier transform of $f_1(x)$ and $f_2(x)$ respectively then prove that:

$$F[a f_1(x) + b f_2(x)] = a F_1(s) + b F_2(s)$$

⇒ Proof :-

By defination

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} . f(t) . dt$$

Given

$$F[f_1(x)] = F_1(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f_1(x) . dx$$

$$F[f_2(x)] = F_2(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f_2(x) . dx$$

$$\therefore F[a f_1(x) + b f_2(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} (a f_1(x) + b f_1(x))$$

$$= a . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} . f_1(x) dx +$$

$$b . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f_2(x) dx$$

$$F[a f_1(x) + b f_2(x)] = a F_1(s) + b F_2(s)$$

Hence proved.

Page No.	
Date	

(b) prove that the fourier Complex integral is

$$f(x) = \frac{1}{2\pi}\int_0^{\infty} e^{-iux}\,du \int_{-\infty}^{\infty} f(t)\, e^{-iut}\,dt$$

⇒ Proof :-

We have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left[2\int_0^{\infty}\cos u(t-x)\,du\right]dt$$

Since,

$$\int_{-\infty}^{\infty}\cos u(t-x)\,du = 2\int_0^{\infty}\cos u(t-x)\,du$$

$$\therefore\ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \times \int_{-\infty}^{\infty}\cos u(t-x)\,du.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\,du\,dt \quad \text{-----} (1)$$

We know that,

$$\int_{-a}^{a} f(t)\,dt = 0 \quad , \text{ when F is odd}$$

$$\therefore\ \int_{-\infty}^{\infty}\sin u(t-x)\,du = 0$$

$$\therefore\ \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t).dt \times \int_{-\infty}^{\infty}\sin u(t-x)\,du = 0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin u(t-x)\,du\,dt = 0 \qquad \text{--- (2)}$$

Adding equation (1) and (2) we get,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\left[\cos u(t-x) + i\sin u(t-x)\right]du\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t).\,e^{-iu(t+x)}\,du\,dt$$

$$\boxed{\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\,du\int_{-\infty}^{\infty} f(t)\,e^{-iut}.dt}$$

Hence proved.

(c) Find the fourier cosine transform of

$$f(x) = e^{-2x} + 4e^{-3x}$$

$\Rightarrow$ Solution :-

Page No.
Date

2018

Q.1

a) Find the laplace transform of $f(t) = e^{at}$.

=> We know that, the laplace transform of $f(t)$ is defined as,

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$\therefore L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at} \cdot dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot dt$$

$$= \left[\frac{e^{-(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= -\frac{1}{s-a} \left[e^{-(s-a)}\right]_0^\infty$$

$$= -\frac{1}{s-a} [e^{-\infty} - e^{0}]$$

$$= -\frac{1}{s-a} [0-1]$$

$$= \frac{1}{s-a}$$

$$\therefore L[e^{at}] = \frac{1}{s-a}$$

b) Show that

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

$\Rightarrow$ let

$$L[u(t-a)] = \int_0^\infty e^{-st} . u(t-a) . dt$$

$$= \int_0^a e^{-st} . 0 \times dt + \int_a^\infty e^{-st} . 1 \times dt$$

$$= 0 + \int_a^\infty e^{-st} . dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_a^\infty$$

$$= -\frac{1}{s} \left[e^{-st} \right]_a^\infty$$

$$= -\frac{1}{s} \left[e^{-\infty} - e^{-as} \right]$$

$$= -\frac{1}{s} \left[0 - e^{-as} \right]$$

$$\boxed{\therefore L[u(t-a) = \frac{e^{-as}}{s}}$$

Page No.			
Date			

(c) Find : $L^{-1}\left[\frac{s}{s^2+16}\right]$

$\Rightarrow$ Let,

$$L^{-1}\left[\frac{s}{s^2+16}\right] = L^{-1}\left[\frac{s}{s^2+4^2}\right]$$

$$= L^{-1}\{L[\cos 4t]\}$$

$$= \cos 4t$$

$$\therefore L^{-1}\left[\frac{s}{s^2+16}\right] = \cos 4t$$

(d) Find the inverse laplace transform of $\frac{3s}{2s+9}$

$\Rightarrow$ Solution :-

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Page No.
Date

(e) Find the fourier sine transform of
$f(x) = 1 \quad 0 < x < a$
$= 0 \quad x > a$

⇒ Solution :-

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cdot \sin st \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a 1 \cdot \sin st \, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos st}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{s} \left[-\cos sa + \cos(0) \right]$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{s} \left[1 - \cos a \right]$$

This is sine transform of $f(x)$.

(f) Define fourier cosine transform of $f(x)$.
⇒